Gauge Theory and Gravitation

Inaugural-Dissertation

submitted

for the Degree of Doctor of Philosophy to the

Philosophical Faculty, section II of the

University of Zurich

by

Martin Andreas Schweizer
from Lampenberg (BL)

Approved by

Prof. Dr. N. Straumann

Zürich 1980

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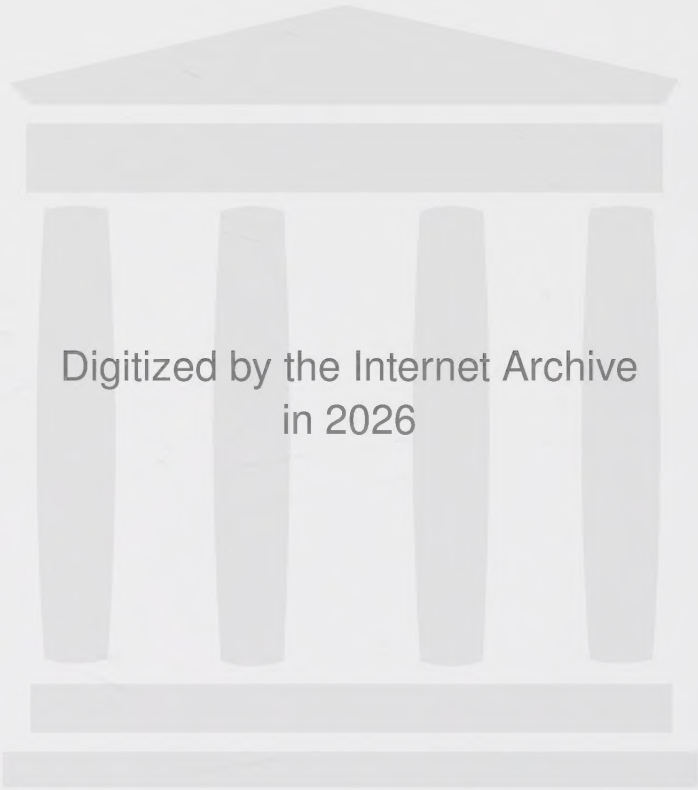
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CONTENTS

I Introduction 1

II Some Mathematical Tools 6

2.1. G-fibre bundles and connections
2.2. Linear frame bundles
2.3. Linear metric connections
2.4. The *-operation
2.5. A variational formula

III Gauge Theory on a Lorentz Manifold 15

3.1. Yang-Mills gauge theory on a curved background
3.2. An alternative U(1)-gauge theory

IV The Gauge Symmetries of Gravitation 22

4.1. The GL(4,R)-invariance
4.2. Local Lorentz invariance and the Cauchy problem
4.3. The Poincaré group as a gauge group
4.4. The diffeomorphism group Diff(M)
4.5. Field equations and the differential identities

V The Levi-Cività Formulation of Gravitation 32

5.1. The relation between $\overset{0}{T}{}^{\mu\nu}$, $t^{\mu\nu}$ and $J^{\lambda\mu\nu}$
5.2. The conserved and symmetric energy-stress tensor $T^{\mu\nu}$
5.3. Table of the Levi-Cività identities
5.4. Field equations
5.5. The Levi-Cività formalism including dynamical torsion

VI Examples of Gravitation Theories 39

6.1. The Utiyama approach
6.2. The Einstein-Cartan theory
6.3. The YM-type Lagrangian for gravitation
6.4. The gauge theory of Fairchild,Jr
6.5. The translational gauge concept of Y.M. Cho
6.6. The gauge concept of M. Carmeli

VII The Poincaré Gauge Theory of Hehl et al. 51

7.1. The teleparallel limit
7.2. The first post-Newtonian approximation
7.3. Conservation laws
7.4. Post-Newtonian generation of gravitational radiation
7.5. Conclusions

VIII Higher Dimensional Unification of Gauge Theory and Gravitation 62

8.1. Heuristic considerations
8.2. Generalization of the Klein-Kaluza theory for non-abelian groups
8.3. Prospects and open questions

Appendix I 71

Appendix II 76

Appendix III 78

Appendix IV 79

Appendix V 80

References 82

I Introduction

The astonishing success of the Glashow-Salam-Weinberg model for the electro-weak interactions has strengthened the belief that the framework of non-abelian gauge theories of Yang & Mills {53} provides the correct description of the basic interactions of nature. The accuracy of the (G-S-W)-model has, however, not yet been tested in the crucial energy-range $E_{cm} \gtrsim 60$ GeV.

Even if the intermediate vector bosons Z, W^+ and W^- should materialize in some accelerator some day, there remains the question about the true nature of the Higgs sector. In the eyes of a critical theorist, this put-in-by-hand mechanism cannot be a satisfactory ultimate concept.

Apart from this mysterious point, the theorists are confronted with the problem of how to incorporate the strong interactions. Up to now there exists no quantum field theory (in four dimensions) other than in terms of a perturbation theory. Thus at present, QCD provides a dynamical quantum field theory for the high momentum range only (asymptotic freedom). It is, however, true that QCD has shown many qualitative successes (spectroscopy, current algebra and chiral symmetry, approximate parton model at high moments, three jet events, consistency with our gauge theories of the weak and electromagnetic interactions, etc.). In spite of the fundamental difficulties it is generally believed that one day

the weak-electromagnetic and the strong interactions will admit a comprehensive description within the framework of a grand unified gauge theory with only one independent coupling constant.

There remains gravity. The final aim is a super unified gauge theory incorporating all the basic interactions of nature. But does gravity admit a description in terms of a gauge theory? And if so, in what sense? These are the basic questions which are discussed in this thesis. Let us briefly summarize the outcome of our analysis:

a) The main dynamical element of gravity, namely the Lorentz metric, is associated with the translation group T^4 and not with the orthochronous Lorentz group $L^{\uparrow}_{+}$. If we gauge the orthochronous Lorentz group $L^{\uparrow}_{+}$, we have to replace the (metric and symmetric) Levi-Cività connection by a dynamical metric connection. But any such metric connection can uniquely be written as a sum of the Levi-Cività connection and a contorsion tensor. Since the Levi-Cività connection is uniquely determined by the associated background metric, the remaining dynamics of the metric connection are described by the contorsion. But the contorsion is a homogeneously transforming tensor. Hence we are invited to reinterpret the contorsion as an additional matter field. We shall see in chapter V that any gravitation theory including dynamical torsion admits an equivalent reformulation in terms of a pure Levi-Cività theory (like general relativity). The dynamics are then given by the matter fields (including torsion) and by a dynamical Lorentz metric. There is nothing left of an $L^{\uparrow}_{+}$ gauge theory, but formally one can still consider the translation

group T^4 as the gauge group of gravitation.

b) Alternatively we consider gravitation theories with dynamical matter fields (excluding torsion), a dynamical Lorentz metric and a dynamical metric connection. This gauge symmetric aspect is discussed in chapter VI. It turns out, however, that $L_+^\uparrow$ should not be a proper internal symmetry group of the total Lagrangian. For instance, we shall find that the YM-type Lagrangian for gravitation which admits $L_+^\uparrow$ as a proper internal symmetry group, provides field equations which have no unique solution of the Cauchy problem. The torsion remains, to a certain extent, undetermined.

The true gauge group is not $L_+^\uparrow$ but rather the inhomogeneous Poincaré group. Any local Lorentz rotations of the fields are then accompanied by local Lorentz rotations of the frames. Formally it is possible to understand gravity as a gauge theory. The metric may be considered as a gauge field associated with the translation group. But T^4 is rather uninteresting as a gauge group, since the action of T^4-valued local gauge transformations on the fields is trivial.

c) In spite of these sobering facts, we believe that there will never be an ultimate unified theory without gravity. In chapter VIII we illustrate that any gauge theory may be considered as a theory on a higher dimensional space. The additional dimensions are "invisible" as a consequence of the "local gauge invariance of physics". In the extended Klein-Kaluza theory the basic gauge fields of nature (the coefficients of the Lorentz metric, the photon , Z, $W^\pm$, etc.) appear as dynamical coefficients of an indefinite metric on such a higher dimensional world.

We investigate also the question of wheter the Higgs fields may find a natural place in this framework. Althoughwe need $L_+^\uparrow$ as a gauge group, it is encouraging to see that a slight modification of the generalized Klein-Kaluza metric may already lead to a symmetry breaking mechanism, which even connects the gravitational coupling constant with the Higgs field and its vacuum expectation value. It would not be surprising if in the end the simplest model should turn out to be the best one since, at present, we see no alternative which is more natural than the Klein-Kaluza unification of gauge theory and gravitation.

In chapter II we develop the theory of principal fibre bundles. Although our presentation is a non-geometrical one, it is equivalent to the higher dimensional bundle-space formulation given in Ref.{27}.

In chapter III the concept of a gauge theory on Minkowski space is generalized to a curved background mainly in terms of the mathematical language developed in the previous chapter. In the second half of chapter III we present an alternative U(1)-gauge theory of Hojman et al.{21} which, by contrast, does not fit into the mathematical framework of chapter II.

In chapter IV we analyse gravitation theories with dynamical matter fields, dynamical Lorentz frames and a dynamical metric connection. In particular we compare the concept of local Lorentz invariance to the concept of local gauge invariance defined in the first part of chapter III. Differential identities are derived as a consequence of the gauge symmetries of the Lagrangian. At the end of chapter IV we give a structure theorem, which connects the differential identities with the field equations associated with the total Lagrangian.

In chapter V we consider gravitation theories with dynamical matter fields and dynamical Lorentz frames. The connection involved is always the Levi-Cività connection and hence is not dynamical. The symmetric and (Levi-Cività) conserved energy-stress tensor is given in terms of the canonical energy-stress tensor and of the spin-density. In particular the gravitation theories of chapter IV are reformulated in this framework. The dynamical torsion appears as an additional matter field.

In chapter VI some examples of gravitation theories are critically discussed. This illustrates the generality of the structures worked out in the chapters IV and V.

In chapter VII the "Poincaré gauge theory" of Hehl et al. {17} is presented in the language of the Cartan calculus. The main results are that the post-Newtonian approximation as well as the post-Newtonian generation of gravitational waves (developed within general relativity by Epstein and Wagoner {9}) are the same as in general relativity. Thus, this alternative theory cannot be distinguished observationally from general relativity.

In chapter VIII we investigate the question whether the Higgs symmetry breaking mechanism follows from a modification of the Klein-Kaluza metric. The gravitational Lagrangian of our model contains the YM-type Lagrangian for gravitation. In the case of Newtonian sources, however, the total Lagrangian reduces to the Lagrangian proposed by Zee {56} and hence the resulting theory is in agreement with observation.

Acknowledgment: I wish to thank Prof. Straumann, as thesis advisor , for his encouragement and his contributions to this work.

II Some Mathematical Tools

It is well known that the underlying mathematical structure of a YM gauge theory finds an adequate interpretation in the geometry of principal fibre bundles. The mathematical sophistication of a rigorous presentation of the theory of connections on principal fibre bundles (see Ref.{27}) exceeds in general the needs of a physicist. Nevertheless we would like to make use of this beautiful mathematical theory. For this reason we give here a rather naive introduction to some basic concepts like tensorforms, connections and curvatures. With the exception of chapter VIII this thesis can be understood with the mathematical apparatus we are going to develop now.

2.1. G-fibre bundles and connections

Let M be an arbitrary differentiable manifold of dimension N , G a linear Lie group and $\mathfrak{g}$ the Lie algebra of G.

Definition: A G-bundle atlas $\mathfrak{A}$ on M is a collection $\{U_j\}_{j\in J}$ of open subsets U_j of M with the following properties:

(i) $\bigcup_{j\in J} U_j = M$

(ii) To every pair $(j_1,j_2)\in J\times J$ with $U_{j_1} \cap U_{j_2} \neq \emptyset$ there is defined a distinct function $g_{j_1 j_2} : U_1 \cap U_2 \longrightarrow G$, such that for all triples $(j_1,j_2,j_3)\in J\times J\times J$ with $U_{j_1} \cap U_{j_2} \cap U_{j_3} \neq \emptyset$ the following composition rule holds

$$g_{j_1 j_2}(x).g_{j_2 j_3}(x) = g_{j_1 j_3}(x) \quad , x \in U_{j_1} \cap U_{j_2} \cap U_{j_3}$$

We say that two G-bundle atlases $\mathfrak{A}$ and $\mathfrak{A}'$ are equivalent , if there exists a G-bundle atlas $\mathfrak{A}''$ such that both , $\mathfrak{A}$ and $\mathfrak{A}'$, are contained in $\mathfrak{A}''$. It is easy to see that this defines an equivalence relation on the set of all G-bundle atlases on M. By the lemma of Zorn every equivalence class of G-bundle atlases has a maximal element $\mathfrak{A}_{max}$.

Definition : A pair $(M,\mathcal{O}_{max})$ is called a principal G-fibre bundle on M . The transition function g_{UV} corresponding to a pair $(U,V)\in\mathcal{O}_{max}\times\mathcal{O}_{max}$, $U\cap V\neq\emptyset$, is called a local gauge transformation.

Let $S:G\longrightarrow Aut(E)$ be a finite dimensional representation of G in a real or complex vector space E .

Definition : A tensor p-form Ψ of type (S,E) on the principal G-fibre bundle $(M,\mathcal{O}_{max})$ is a collection $\{\Psi_U\}_{U\in\mathcal{O}_{max}}$ of E-valued p-forms with the following properties:

(i) For any $U\in\mathcal{O}_{max}$ the form Ψ_U is defined on U.

(ii) For every pair $(U,V)\in\mathcal{O}_{max}\times\mathcal{O}_{max}$ with $U\cap V\neq\emptyset$ the following holds

$$\Psi_U(x) = S(g_{UV}(x)).\Psi_V(x) \tag{2.1}$$

Let $\theta^1,\ldots,\theta^N$ be a local frame of 1-forms on some $U\in\mathcal{O}_{max}$. We may expand Ψ_U relative to this frame

$$\Psi_U = \sum_{i_1<\ldots<i_p} \Psi_U{}_{i_1\ldots i_p}\ \theta^{i_1}\wedge\ldots\wedge\theta^{i_p} \tag{2.2}$$

Furthermore let $X_1,\ldots,X_K$ denote a basis of E

$$\Psi_U\ {}_{i_1\ldots i_p} = \Sigma_{a=1}^{K}\ \Psi^a_{U i_1\ldots i_p}\ X_a \tag{2.3}$$

We notice that Ψ_U carries two types of indices :

(i) external indices $(i_1,\ldots,i_p)$ transforming like usual covariant tensor indices.

(ii) internal indices (a) transforming according to the group representation $S^a_b(g_{UV}(x))$.

Internal and external transformations are independent one of each other.

Definition : A connection or a gauge potential A on the principal G-fibre bundle $(M,\mathcal{O}_{max})$ is a collection $\{A_U\}_{U\in\mathcal{O}_{max}}$ of locally defined $\mathfrak{g}$-valued 1-forms A_U subject to the transformation rule:

$$A_U(x) = g_{UV}(x)A_V(x)\bar{g}^{-1}_{UV}(x) - dg_{UV}(x).\bar{g}^{-1}_{UV}(x) \tag{2.4}$$

$x\in U\cap V\neq\emptyset$, $(U,V)\in\mathcal{O}_{max}\times\mathcal{O}_{max}$.

Notice that up to the inhomogeneous Cartan term $-dg_{UV}\bar{g}^{-1}_{UV}$ a connection A transforms like a tensor 1-form of type $(Ad_G, \mathcal{G})$.

The exterior covariant derivative of a tensor p-form Ψ of type (S,E) is defined as follows :

$$D\Psi_U = d\Psi_U + S_*(A_U)\wedge\Psi_U \quad ; \quad U\in\mathcal{O}_{max} \tag{2.5}$$

Hereby S_* denotes the infinitesimal representation $\mathcal{G} \longrightarrow End(E)$ induced by S, and $S_*(A_U)\wedge\Psi_U = A_U^a\wedge\Psi_U^b\, S_*(T_a).X_b$; T_a is a basis of $\mathcal{G}$. Actually the collection $\{D\Psi_U\}_{U\in\mathcal{O}_{max}}$ is a tensor (p+1)-form $D\Psi$ of type (S,E). The proof of this is straightforward.

Since the connection is subject to an inhomogeneous transformation rule, a separate definition must be given for the covariant derivative of A .We define

$$F_U = DA_U = dA_U + 1/2\, A_U^a\wedge A_U^b [T_a,T_b] \tag{2.6}$$

It is easy to see that the collection $\{F_U\}_{U\in\mathcal{O}_{max}}$ defines a tensor 2-form F = DA of type $(Ad_G, \mathcal{G})$. We call F the curvature or the YM field strength belonging to A.

The curvature satisfies the Bianchi identity

$$DF = 0 \tag{2.7}$$

It is easy to see that on Minkowski space equation (2.7) reads as follows

$$F_{[\mu\nu,\lambda]} + \left[A_{[\mu}\,;\, F_{\nu\lambda]}\right] = 0 \tag{2.7'}$$

For later use we give here the formula for the second exterior covariant derivative of a tensor p-form Ψ

$$D^2\Psi = S_*(F)\wedge\Psi \tag{2.8}$$

2.2. Linear frame bundles

Let now G be the general linear group $GL(N,\mathbb{R})$. Then any $U\in\mathcal{O}_{max}$ admits a unique frame $\theta_U^1,\ldots,\theta_U^N$ on U , such that

$$\theta_U^i(x) = g_{UV}{}^i_j(x).\theta_V^j(x) \tag{2.9}$$

for all $(U,V)\in\mathcal{O}_{max}\times\mathcal{O}_{max}$; $U\cap V \neq \emptyset$.

From (2.9) it is obvious that the collection $\{\theta_U\}_{U\in\mathcal{A}_{max}}$ of $\mathbb{R}^N$-valued 1-forms

$$\theta_U = \Sigma_{j=1}^N \theta_U^j e_j \tag{2.10}$$

(e_j: canonical basis in $\mathbb{R}^N$) defines a tensor 1-form θ of type $(id_{GL(N,\mathbb{R})};\mathbb{R}^N)$ on $(M,\mathcal{A}_{max})$.

The appearance of this 1-form θ , called the translational gauge potential , is a peculiarity of $GL(N,\mathbb{R})$ and of some of its subgroups. Since the group acts on the frames as well, the corresponding bundles are called linear frame bundles. In this case we denote a connection by ω and the corresponding curvature by Ω.

Let E_j^i denote the canonical basis of $gl(N,\mathbb{R})$, $(E_j^i)_s^r = \delta^{ir}\,\delta_{js}$, and let U be an element of $\mathcal{A}_{max}$.Then we can introduce the following expansions:

$$\begin{aligned}
\omega_U &= \omega_j^i\, E_i^j \\
\omega_j^i &= \Gamma_{aj}^{\ \ i}\theta_U^a \\
\Omega_U &= \Omega_j^i\, E_i^j \\
\Omega_j^i &= 1/2\ R_{jab}^i\ \theta_U^a\wedge\theta_U^b
\end{aligned} \tag{2.11}$$

If we express (2.4) in terms of the components of ω then we obtain the usual transformation rule for the components of the Christoffel symbols relative to arbitrary frames.

Notice that tensorfields of type $T_s^r(M)$ can be interpreted as tensor o-forms of type $(T_s^r,\ L_s^r)$, where

$$L_s^r = \underbrace{\mathbb{R}^N\otimes\ \ldots.\ \otimes\mathbb{R}^N}_{r}\otimes\underbrace{(\mathbb{R}^N)^*\otimes\ldots.\otimes(\mathbb{R}^N)^*}_{s}$$

More generally, let T be a tensor p-form of type (T_s^r, L_s^r). From (2.5) one finds

$$(DT)^{i_1\ldots i_r}_{j_1\ldots j_s} = dT^{i_1\ldots i_r}_{j_1\ldots j_s} + \Sigma_h\,\omega_f^{i_h}\wedge T^{i_1\ldots\overset{\vee}{f}\ldots i_r}_{j_1\ldots\ldots j_s} - \Sigma_h\,\omega_{j_h}^f\wedge T^{i_1\ldots\ldots i_r}_{j_1\ldots\hat{f}\ldots j_s} \tag{2.12}$$

The exterior covariant derivative

$$\Theta = D\theta \tag{2.13}$$

of θ is called the torsion .

Remark : Given a tensor p-form $T_{Uj_1\ldots j_s}^{i_1\ldots i_r}$ on some $U\in\mathcal{O}_{max}$, we can expand it relative to the frame θ_U^i , i.e.

$$T_{Uj_1\ldots j_s}^{i_1\ldots i_r} = \sum_{r_1<\ldots<r_p} T_{U\ r_1\ldots r_p}^{\ \ i_1\ldots i_r}{}_{j_1\ldots j_s}\ \theta_U^{r_1}\wedge\ldots\wedge\theta_U^{r_p} \qquad (2.14)$$

The indices $(r_1,\ldots,r_p)$ are external indices.But since the group $GL(N,\mathbb{R})$, acting on the internal indices $(^{i_1\ldots i_r}_{j_1\ldots j_s})$, operates also on the frames θ_U^r , we are free to reinterpret $(r_1,\ldots,r_p)$ as covariant internal tensor indices.

2.3. Linear metric connections

For the rest of this chapter let (M,g) be an orientable Lorentz manifold. The signature of g is taken as (1,-1,-1,-1). We are mainly interested in metric connections ω , i.e. linear connections with the property

$$Dg_{\mu\nu} = 0 \qquad (2.15)$$
$$(g = g_{\mu\nu}\,\theta^\mu\otimes\theta^\nu)$$

Equation (2.15) implies

$$D\eta_{\mu\nu\lambda\sigma} = 0 \qquad (2.16)$$

where

$$\eta_{\mu\nu\lambda\sigma} = \sqrt{-g}\,\varepsilon_{\mu\nu\lambda\sigma} \qquad (2.17)$$

$\eta_{\mu\nu\lambda\sigma}$ is a tensor o-form of type (T_4^0,L_4^0). As in Ref.{46} we introduce the following tensor forms of higher degree

$$\begin{aligned}
\eta_{\mu\nu\lambda} &= \theta^\sigma\ \eta_{\mu\nu\lambda\sigma}\\
\eta_{\mu\nu} &= 1/2\ \ \theta^\lambda\wedge\eta_{\mu\nu\lambda}\\
\eta_{\mu} &= 1/3\ \ \theta^\nu\wedge\eta_{\mu\nu}\\
\eta &= 1/4\ \ \theta^\mu\wedge\eta_{\mu}
\end{aligned} \qquad (2.18)$$

The following list of algebraic identities will be very useful. Symmetrizing and antisymmetrizing brackets are always understood without combinatoric factors.

$$\theta^\rho \wedge \eta_{\mu\nu\lambda\sigma} = 1/3!\,\delta^\rho_{[\sigma}\,\eta_{\mu\nu\lambda]}$$
$$\theta^\rho \wedge \eta_{\mu\nu\lambda} = 1/2!\,\delta^\rho_{[\lambda}\,\eta_{\mu\nu]}$$
$$\theta^\rho \wedge \eta_{\mu\nu} = \delta^\rho_{[\nu}\,\eta_{\mu]}$$
$$\theta^\rho \wedge \eta_{\mu} = \delta^\rho_{\mu}\,\eta \qquad (2.19)$$

The torsion is a tensor 2-form of type $(T^1_0, \mathbb{R}^4)$

$$\Theta^\mu = 1/2\ Q^\mu_{\alpha\beta}\ \theta^\alpha \wedge \theta^\beta \qquad (2.20)$$

We give all the covariant derivatives of the forms in (2.18) . The connection is supposed to be metric :

$$D\eta_{\mu\nu\lambda} = \Theta^\sigma \eta_{\mu\nu\lambda\sigma}$$
$$D\eta_{\mu\nu} = \Theta^\lambda \wedge \eta_{\mu\nu\lambda} = (Q^\lambda_{\mu\nu} + \delta^\lambda_\mu \cdot Q^\sigma_{\nu\sigma} - \delta^\lambda_\nu \cdot Q^\sigma_{\mu\sigma})\,\eta_\lambda$$
$$D\eta_{\mu} = \Theta^\nu \wedge \eta_{\mu\nu} = Q^\nu_{\mu\nu}\,\eta \qquad (2.21)$$

A quadruple $(\xi_0, \xi_1, \xi_2, \xi_3)$ of locally defined vector fields is called a Lorentz frame , if

$$g(\xi_\mu, \xi_\nu) = \hat{\eta}_{\mu\nu} = \begin{bmatrix} 1 & \\ & -1_3 \end{bmatrix}$$

The corresponding dual frame θ^μ ($\theta^\mu(\xi_\nu) = \delta^\mu_\nu$) is called a Lorentz frame , too. All Lorentz frames are supposed to have the same orientation. Expressing (2.6) , (2.13) and (2.15) in terms of the components relative to the internal space yields the Cartan structure equations

(i) $\quad dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$

(ii) $\quad \Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu$

(iii) $\quad \Theta^\mu = d\theta^\mu + \omega^\mu{}_\lambda \wedge \theta^\lambda \qquad (2.22)$

In a Lorentz frame we have $dg_{\mu\nu} = 0$ and consequently

(i') $\quad 0 = \omega_{\mu\nu} + \omega_{\nu\mu} \qquad (2.23)$

This amounts to the statement that ω takes its values in the Lie subalgebra so(1,3). A natural basis of so(1,3) is given by the matrices

$$B_\alpha{}^\beta \qquad (\alpha, \beta = 0,1,2,3)$$
$$(B^{\alpha\beta})^\mu{}_\nu = \hat{\eta}^{\alpha\mu}\delta^\beta_\nu - \hat{\eta}^{\beta\mu}\delta^\alpha_\nu \qquad (2.24)$$

We restrict ourselves to Lorentz frames and expand ω with respect to this smaller basis:

$$\omega = 1/2\ \omega^{\alpha}{}_{\beta}\ B_{\alpha}{}^{\beta} = 1/2\ \omega_{\alpha\beta}\ B^{\alpha\beta} \tag{2.25}$$

Let $\Lambda: SL(2,\mathbb{C}) \longrightarrow L_+^{\uparrow}$ denote the covering homomorphism onto the orthochronous Lorentz group and let $\Lambda_*: sl(2,\mathbb{C}) \longrightarrow so(1,3)$ denote its differential. Then we obtain a natural basis of $sl(2,\mathbb{C})$ by putting

$$B_{\alpha}{}^{\beta} = \bar{\Lambda}_*^{-1}\ B_{\alpha}{}^{\beta} \tag{2.26}$$

With $\mathcal{Q}^{\wedge}$ we denote the collection of all $U \in \mathcal{Q}_{max}$ with the property that $\theta_U^0, \theta_U^1, \theta_U^2, \theta_U^3$ is a Lorentz frame. For a pair $(U,V) \in \mathcal{Q}_{max} \times \mathcal{Q}_{max}$, $U \cap V \neq \emptyset$, the corresponding local gauge transformation g_{UV} takes its values in $L_+^{\uparrow}$. Hence, by the maximality of $\mathcal{Q}^{\wedge}$, we obtain a $L_+^{\uparrow}$-fibre bundle $(\mathcal{Q}^{\wedge}, M)$ called the Lorentz frame bundle of the metric g. It is obvious from (2.25) that any metric connection on $(\mathcal{Q}_{max}, M)$ defines a connection on $(\mathcal{Q}^{\wedge}, M)$.

Let $(\mathcal{Q}^{SL}, M)$ be a $SL(2,\mathbb{C})$-fibre bundle over M. For any pair $(U,V) \in \mathcal{Q}^{SL} \times \mathcal{Q}^{SL}$, $U \cap V \neq \emptyset$, and its associated local gauge transformation $g_{UV}(x)$ the function $\Lambda(g_{UV}(x))$ takes its values in $L_+^{\uparrow}$. In this way one obtains a $L_+^{\uparrow}$-fibre bundle $(\Lambda(\mathcal{Q}^{SL}), M)$.

We say that $(\mathcal{Q}^{SL}, M)$ is a covering bundle of a Lorentz frame bundle $(\mathcal{Q}^{\wedge}, M)$ if $(\Lambda(\mathcal{Q}^{SL}), M)$ coincides with $(\mathcal{Q}^{\wedge}, M)$. The existence of a covering bundle is guaranted if there is a $U \in \mathcal{Q}^{\wedge}$, U equal to M as a point set (see Ref.{12}).

Apart from global problems it is clear how to incorporate spinor fields. In addition (2.25) and (2.26) allow us to reinterpret any metric connection as a $sl(2,\mathbb{C})$-valued spinor connection. The exterior covariant derivative of a spinor p-form with respect to a metric connection is immediately obtained from (2.5).

2.4. The *-operation

The *-operator associated with the metric of a Lorentz manifold (M,g) assigns to each p-form $\alpha \in \Lambda^p(M)$ a (4-p)-form $*\alpha \in \Lambda^{4-p}(M)$. Relative to a local frame it is defined as follows:

$$\begin{aligned} *1 &= \eta \\ *\theta^{\mu} &= \eta^{\mu} \\ *(\theta^{\mu} \wedge \theta^{\nu}) &= \eta^{\mu\nu} \\ *(\theta^{\mu} \wedge \theta^{\nu} \wedge \theta^{\lambda}) &= \eta^{\mu\nu\lambda} \\ *(\theta^{\mu} \wedge \theta^{\nu} \wedge \theta^{\lambda} \wedge \theta^{\sigma}) &= \eta^{\mu\nu\lambda\sigma} \end{aligned} \tag{2.27}$$

The inverse of the *-operation is given by the formula

$$**\alpha = (-1)^{p(4-p)+3}\,\alpha \tag{2.28}$$

We apply (2.28) to (2.27) and obtain

$$\begin{aligned} *\eta &= -1 \\ *\eta^{\mu} &= \theta^{\mu} \\ *\eta^{\mu\nu} &= -\theta^{\mu}\wedge\theta^{\nu} \\ *\eta^{\mu\nu\lambda} &= \theta^{\mu}\wedge\theta^{\nu}\wedge\theta^{\lambda} \\ *\eta^{\mu\nu\lambda\sigma} &= -\theta^{\mu}\wedge\theta^{\nu}\wedge\theta^{\lambda}\wedge\theta^{\sigma} \end{aligned} \tag{2.29}$$

2.5. A variational formula

Given a Lorentz manifold (M,g) then there exists a unique linear metric connection ${}^{LC}\omega$ on M with vanishing torsion. This is the so-called Levi-Cività connection.

Relative to a Lorentz frame the Cartan structure equations (2.22) read as follows

$$\begin{aligned} &\text{(i)} && {}^{LC}\omega_{\mu\nu} + {}^{LC}\omega_{\nu\mu} = 0 \\ &\text{(ii)} && d{}^{LC}\omega^{\mu}{}_{\nu} + {}^{LC}\omega^{\mu}{}_{\lambda}\wedge{}^{LC}\omega^{\lambda}{}_{\nu} = {}^{LC}\Omega^{\mu}{}_{\nu} \\ &\text{(iii)} && d\theta^{\mu} + {}^{LC}\omega^{\mu}{}_{\lambda}\wedge\theta^{\lambda} = 0 \end{aligned} \tag{2.30}$$

The solution of (2.30) is given in section 6.5. below .

Now let $\theta^{\mu}(x)$ be a local Lorentz frame and $\theta^{\mu}(x,s)$ a differentiable one-parameter family of locally defined frames , such that $\theta^{\mu}(x,0) = \theta^{\mu}(x)$. The parameter s is supposed to run through the points of an open interval around zero. Any s defines a (local) Lorentz metric

$$g(x,s) = \hat{\eta}_{\mu\nu}\;\theta^{\mu}(x,s)\otimes\theta^{\nu}(x,s) \tag{2.31}$$

In addition a unique linear metric connection ${}^{LC}\omega^{\mu}{}_{\nu}(x,s)$ is associated with g(x,s) :

$$\begin{aligned} &\text{(i)} && {}^{LC}\omega_{\mu\nu}(x,s) + {}^{LC}\omega_{\nu\mu}(x,s) = 0 \\ &\text{(ii)} && d\theta^{\mu}(x,s) + {}^{LC}\omega^{\mu}{}_{\lambda}(x,s)\wedge\theta^{\lambda}(x,s) = 0 \end{aligned} \tag{2.32}$$

The variation $\delta\theta^{\mu}(x) = \frac{d}{ds}|_{o}\theta^{\mu}(x,s)$ of $\theta^{\mu}(x)$ induces a unique variation $\delta\overset{LC}{\omega}_{\mu\nu}(x)$ of $\overset{LC}{\omega}_{\mu\nu}(x)$ determined by

(i') $$\delta\overset{LC}{\omega}_{\mu\nu}(x) + \delta\overset{LC}{\omega}_{\nu\mu}(x) = 0 \qquad (2.33)$$

(ii') $$\overset{LC}{D}\delta\theta^{\mu} + \delta\overset{LC}{\omega}{}^{\mu}{}_{\lambda}\wedge\theta^{\lambda} = 0 \qquad (2.34)$$

The unique solution of these equations is

$$\delta\overset{LC}{\omega}_{\mu\nu} = -1/2\theta^{\lambda}.*\{\overset{LC}{D}(\delta\theta_{\mu}\wedge\eta_{\nu\lambda} - \delta\theta_{\nu}\wedge\eta_{\mu\lambda} - \delta\theta_{\lambda}\wedge\eta_{\mu\nu})\} \qquad (2.35)$$

The antisymmetry (2.33) is satisfied. We show that (2.34) holds,too. For this purpose we contract (2.35) with θ^{ν} and multiply the product with $\eta_{\alpha\beta}$:

$$-1/2\ *\{\overset{LC}{D}(\delta\theta_{\mu}\wedge\eta_{\nu\lambda} - \delta\theta_{\nu}\wedge\eta_{\mu\lambda} - \delta\theta_{\lambda}\wedge\eta_{\mu\nu})\}.\theta^{\lambda}\wedge\theta^{\nu}\wedge\eta_{\alpha\beta} =$$

$$= -1/2\ \overset{LC}{D}(\delta\theta_{\mu}\wedge\eta_{\nu\lambda} - \delta\theta_{\nu}\wedge\eta_{\mu\lambda} - \delta\theta_{\lambda}\wedge\eta_{\mu\nu})(\delta^{\nu}_{\beta}\,\delta^{\lambda}_{\alpha} - \delta^{\nu}_{\alpha}\,\delta^{\lambda}_{\beta})\ *\eta$$

$$= -\overset{LC}{D}(\delta\theta_{\mu}\wedge\eta_{\alpha\beta}) = -\overset{LC}{D}\delta\theta_{\mu}\wedge\eta_{\alpha\beta}\ ;$$

Since this holds for all $\eta_{\alpha\beta}$, we are done .

III Gauge Theory on a Lorentz Manifold [35]

With the help of the mathematical apparatus developed in the previous chapter it is straightforward to generalize the concept of a gauge theory on Minkowski space to a curved background. This is done in section 3.1. In the subsequent chapter IV we shall analyse the gauge symmetries of Lagrangian field theories of gravitation. A comparison with section 3.1. will elucidate the peculiarities of the gauge structure of gravity.

In section 3.2. we shall consider an alternative U(1)-gauge theory, which does not fit into the mathematical framework of chapter II. In this modified theory the gauge potential couples to a dynamical background torsion. (see also Ref.{21})

3.1. Yang-Mills gauge theory on a curved background

a) Local gauge invariance

Let G be the structure group of a gauge theory. In the (G-S-W)-model G is equal to $SU(2)\times U(1)$.

The matter fields Ψ are taken to be tensor p-forms of type (S,E) on a G-fibre bundle $(\mathcal{Q}_{max}, M)$. The YM potential A is a connection on this bundle.

Let $\theta^0,\theta^1,\theta^2,\theta^3$ be a local frame and $(T_a)_{a=1,\ldots,K}$ a basis of the Lie algebra $\mathfrak{g}$ of G. We use the decomposition

$$\begin{aligned} A &= A_\mu\, \theta^\mu \\ A &= A^a_\mu\, T_a \end{aligned} \qquad (3.1)$$

The YM field strength is the curvature F belonging to A

$$F = 1/2\, F_{\mu\nu}\, \theta^\mu \wedge \theta^\nu = dA + 1/2\, A^a \wedge A^b\, [T_a, T_b] \qquad (3.2)$$

The canonical scalarproduct $K: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} K(X,Y) &= -\,\mathrm{Tr}(\mathrm{ad}(X).\mathrm{ad}(Y)) \\ \mathrm{ad}(X)Z &= [X,Z] \quad ; X,Y,Z \in \mathfrak{g} \end{aligned} \qquad (3.3)$$

is the so-called Killing form on $\mathfrak{g}$.

With the minus sign in (3.3) the Killing form of the Lie algebra of a semisimple compact group is positive definite.

It is easy to see that

$$K([X,Y],Z) = K(X,[Y,Z]) \tag{3.4}$$

For arbitrary $\mathfrak{g}$-valued differential forms α and β we define

$$K(\alpha,\beta) = \alpha\wedge\beta = \alpha^a\wedge\beta^b\, K(T_a,T_b) \tag{3.5}$$

An example is the Lagrangian density

$$L_F = -1/2\, K(F,*F) = -1/2\, F\wedge *F \tag{3.6}$$

of the free YM-field.

Definition: A Lagrangian density $L(F,\Psi,D\Psi)$ is called locally gauge invariant, if for all local gauge transformations $g(x)\in G$ the following holds (up to an exact differential)

$$L(F(x),\Psi(x),D\Psi(x)) = L(Ad_G(g(x))F(x),S(g(x))\Psi(x),S(g(x))D\Psi(x)) \tag{3.7}$$

Since Ad_G leaves K invariant (3.7) is satisfied for the density (3.6).

b) The YM field equations

The Lagrangian leading to the coupled YM equations is

$$L_{YM} = -1/2\, F\wedge *F + L(\Psi,D\Psi) \tag{3.8}$$

Let D be a compact region with regular boundary. The field equations follow from variations of the action

$$I = \int_D L_{YM} \tag{3.9}$$

with respect to A.

Since $1/2\,\delta(F\wedge *F) = \delta F\wedge *F$ and $\delta F = d\delta A + [\delta A,A]$ we find $\delta F\wedge *F = d\delta A\wedge *F + K([\delta A,A],*F)$. An integration by parts and (3.4) yield

$$\delta F\wedge *F = \delta A\wedge D*F + \text{exact differential} \tag{3.10}$$

Furthermore there is a tensor 3-form J of type $(Ad_G,\mathfrak{g})$, called the YM-current ($J = J^\mu\cdot\eta_\mu$; $J^\mu = J^{a\mu}\cdot T_a$), such that

$$\delta L(\Psi,D\Psi) = \delta A\wedge J + \text{exact differential} \tag{3.11}$$

Taken (3.10) and (3.11) together gives

$$\delta I = - \int_D K(\delta A, D^*F - J) + \text{exact differential} \qquad (3.12)$$

The variations are supposed to vanish on M-D . We extremize the action (3.9) for such variations and find

$$K(\delta A, D^*F - J) = 0 \qquad (3.13)$$

For semisimple Lie algebras the Killing form K is nondegenerate and (3.13) yields the coupled YM equations

$$D^*F = J \qquad (3.14)$$

In the model of Glashow, Salam and Weinberg the Lie algebra is not semisimple. But in this case it is easy to introduce a nondegenerate scalar product K having all the desired properties.

c) Differential identities

The exterior covariant derivative of the left hand side of (3.14) vanishes identically : From formula (2.8) we find

$$DD^*F = \mathrm{ad}(F) \wedge {}^*F = F^a \wedge {}^*F^b \left[T_a, T_b\right] = 0$$

The equations

$$\text{(a)} \quad DD^*F = 0 \qquad \text{(b)} \quad DJ = 0 \qquad (3.15)$$

are differential identities in the sense that they are consequences of the local gauge invariance of L_F and of $L(\Psi, D\Psi)$, respectively.

In order to prove this let D be a region like above and $h: M \longrightarrow \mathfrak{g}$ a function , h = 0 on M-D. Then $g(x,s) = \exp(s.h(x))$, $s \in \mathbb{R}$, is a one-parameter group of local gauge transformations:

$$\Psi'(x,s) = S(\exp(sh(x)))\, \Psi(x)$$

$$A'(x,s) = \mathrm{Ad}_G(\exp(sh(x)))\, A(x) - d(e^{sh(x)}).e^{-sh(x)}$$

The induced infinitesimal gauge transformations are

$$\delta\Psi(x) = \frac{\partial}{\partial s}\Big|_0 \Psi'(x,s) = S_*(h(x)).\Psi(x)$$

$$\delta A(x) = \frac{\partial}{\partial s}\Big|_0 A'(x,s) = -Dh(x)$$

The action integrals $I_\Psi = \int_D L(\Psi'(x,s), D'\Psi'(x,s))$

and

$$I_A = \int_D F'(x,s) \wedge {}^*F'(x,s)$$

are independent of s.

Differentiating these actions with respect to s and substituting -Dh for δA one finds , using the equations for the matter fields:

$$0 = \int_D Dh \wedge J = -\int_D h.DJ$$

and

$$0 = \int_D Dh \wedge D^*F = -\int_D h.DD^*F$$

Since this holds for arbitrary functions h of the above specified type, our claim is proven.

d) Component expressions

Let us compute the component expressions of (3.14). From (2.5) we we have

$$D^*F = d^*F + ad(A) \wedge {}^*F \tag{3.16}$$

The first term can be written with respect to a local frame as follows

$$\begin{aligned} d^*F &= 1/2\, d(F^{\mu\nu}\, \eta_{\mu\nu}) \\ &= 1/2\, \{dF^{\mu\nu} + F^{\alpha\beta}\, (\overset{LC}{\omega}{}^{\mu}{}_{\alpha} + \overset{LC}{\omega}{}^{\nu}{}_{\beta})\} \wedge \eta_{\mu\nu} \\ &= 1/2\, \overset{LC}{D} F^{\mu\nu} \wedge \eta_{\mu\nu} \\ &= 1/2\, F^{\mu\nu}{}_{|\lambda}\ \theta^{\lambda} \wedge \eta_{\mu\nu} = F^{\mu\nu}{}_{|\nu}\ \eta_{\mu} \end{aligned} \tag{3.17}$$

Here $F^{\mu\nu}{}_{|\lambda}$ denotes the covariant derivative of the tensor indices with respect to the Levi-Civitá connection.

The second term in (3.16) is

$$ad(A) \wedge {}^*F = 1/2 \left[A_{\lambda}, F^{\mu\nu}\right] \theta^{\lambda} \wedge \eta_{\mu\nu} = \left[A_{\nu}, F^{\mu\nu}\right] \eta_{\mu} \tag{3.18}$$

Now we consider the case that the connection of the gravitational background is some metric connection $\omega^{\alpha}{}_{\beta}$, not $\overset{LC}{\omega}{}^{\alpha}{}_{\beta}$. Let $Q^{\alpha}_{\mu\nu}$ denote the components of the associated torsion.Noticing that (2.21) implies

$$d\eta_{\mu\nu} = \omega_{\mu}{}^{\lambda} \wedge \eta_{\lambda\nu} + \omega^{\rho}{}_{\nu} \wedge \eta_{\mu\rho} + (Q^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu} Q^{\sigma}_{\nu\sigma} - \delta^{\lambda}_{\nu} Q^{\sigma}_{\mu\sigma})\, \eta_{\lambda}$$

allows us to repeat the computation (3.17). We find

$$d^*F = F^{\mu\nu}{}_{;\nu} \eta_{\mu} + 1/2\, F^{\sigma\nu} (Q^{\mu}_{\sigma\nu} + \delta^{\mu}_{\sigma} Q^{\lambda}_{\nu\lambda} - \delta^{\mu}_{\nu} Q^{\lambda}_{\sigma\lambda}) \eta_{\mu}$$

Now $F^{\mu\nu}{}_{;\lambda}$ denotes the covariant derivative with respect to $\omega^{\alpha}{}_{\beta}$.

Similar computations can be performed for the identities (3.15). Taken all together gives the component expressions of the

YM equations:

$$F^{\mu\nu}{}_{|\nu} + \left[A_{\nu}, F^{\mu\nu}\right] = J^{\mu} \tag{3.19}$$

or alternatively

$$F^{\mu\nu}{}_{;\nu} + \left[A_{\nu}, F^{\mu\nu}\right] + 1/2\, F^{\sigma\nu} (Q^{\mu}_{\sigma\nu} + \delta^{\mu}_{\sigma}\ Q^{\lambda}_{\nu\lambda} - \delta^{\mu}_{\nu}\, Q^{\lambda}_{\sigma\lambda}) = J^{\mu} \tag{3.20}$$

3.2. An alternative U(1)-gauge theory

Notice that torsion in (3.20) appears just spuriously and does not couple! This reflects the fact that the *-operator in (3.14) is determined by the metric alone.

Nevertheless, Hojman et al. {21} have proposed a modified U(1)-gauge theory, where torsion actually does come into play. With arguments we are not going to repeat here, they consider a special type of torsion $Q^{\alpha}_{\mu\nu}$ given by a scalar potential Φ in the following way (In units $\hbar = c = 1$ the potential Φ is dimensionless.):

$$Q^{\alpha}_{\mu\nu} = \delta^{\alpha}_{\nu}\,\Phi_{,\mu} - \delta^{\alpha}_{\mu}\,\Phi_{,\nu} \tag{3.21}$$

(In this section we work exclusively in local coordinates.)

The local gauge transformation for a charged scalar field Ψ is the usual one

$$\Psi(x) \longrightarrow \Psi'(x) = e^{iq\Lambda(x)}\,\Psi(x) \tag{3.22}$$

The rule for the local gauge transformation of the vector potential is, however, a modified one

$$A_{\mu}(x) \longrightarrow A'_{\mu}(x) = A_{\mu}(x) + e^{\Phi(x)}\,\Lambda_{,\mu}(x) \tag{3.23}$$

Clearly, the Lagrangian

$$\mathcal{L}_{\Psi} = -\frac{1}{4\Pi}\,(D_{\mu}\Psi^{*})(D^{\mu}\Psi)\,\sqrt{-g} \tag{3.24}$$

is locally gauge invariant with respect to (3.22) and (3.23) if the covariant derivative is taken to be

$$\begin{aligned} D_{\mu}\Psi &= \partial_{\mu}\Psi - iq\,e^{-\Phi}\,A_{\mu}\,\Psi \\ D_{\mu}\Psi^{*} &= \partial_{\mu}\Psi^{*} + iq\,e^{-\Phi}\,A_{\mu}\Psi^{*} \end{aligned} \tag{3.25}$$

In order to achieve that the Lagrangian

$$\mathcal{L}_{A} = -\frac{1}{16\Pi}\,F_{\mu\nu}F^{\mu\nu}\,\sqrt{-g} \tag{3.26}$$

is locally gauge invariant with respect to (3.23), the definition of the YM field strength must be modified:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} + A_{\mu}\,\Phi_{,\nu} - A_{\nu}\,\Phi_{,\mu} \tag{3.27}$$

Notice that (3.23) and (3.27) imply

$$F'_{\mu\nu} = F_{\mu\nu} \tag{3.28}$$

The dynamical fields incorporated in the total Lagrangian are

charged scalar field Ψ , the vector potential A_μ , the metric tensor $g_{\mu\nu}$ and the scalar potential Φ :

$$\mathcal{L}_{tot} = \frac{1}{4\Pi}\{R - F_{\mu\nu}F^{\mu\nu} - 4\, D_\mu\Psi^*\, D^\mu\Psi\}\sqrt{-g} \tag{3.29}$$

(R denotes the curvature scalar associated with the metric connection including the torsion $Q^\alpha_{\mu\nu}$.)

A generalized version of (3.29) is discussed in Ref.{22}. Up to a divergence one has

$$R = \overset{LC}{R} + 1/2\; Q^\alpha_{\rho\tau}Q^{\tau\rho}_{\ \ \alpha} + 1/4\; Q^\alpha_{\rho\tau}Q_\alpha^{\ \rho\tau} - Q^\alpha_{\rho\alpha}Q^{\sigma\rho}_{\ \ \sigma} \tag{3.30}$$

For the specific type of torsion (3.21) the Lagrangian (3.29) takes the form

$$\mathcal{L}_{tot} = \frac{1}{4\Pi}\{\overset{LC}{R} - 6\Phi_{,\alpha}\,\Phi'^{,\alpha} - F_{\mu\nu}F^{\mu\nu} - 4\, D_\mu\Psi^*\, D^\mu\Psi\}\sqrt{-g} \tag{3.31}$$

In contrast to the Einstein-Cartan theory (to be discussed in chapter VI) torsion in this theory is dynamical and, in particular, interacts with the photon field. More explicitly , we consider the field equations resulting from variations with respect to A_μ and Φ

$$F^{\mu\nu}{}_{|\nu} + F^{\mu\nu}\Phi_{|\nu} = iq\, e^{-\Phi}(D^\mu\Psi^*.\Psi - D^\mu\Psi.\Psi^*) \tag{3.32}$$

$$3\Phi^{|\alpha}{}_{|\alpha} + (F^{\mu\nu}A_\mu)_{|\nu} + iq\, e^{-\Phi}A_\mu(D^\mu\Psi.\Psi^* - D^\mu\Psi^*.\Psi) = 0 \tag{3.33}$$

These two equations imply

$$\Phi^{|\alpha}{}_{|\alpha} = 1/6\; F_{\mu\nu}F^{\mu\nu} \tag{3.34}$$

The electromagnetic field generates torsion, and for $F_{\mu\nu} = 0$ the scalar field Φ obeys a wave equation. Hence torsion can propagate in vacuum. Notice that the electromagnetic source term in (3.34) includes terms quadratic in Φ. This nonlinearity disappears,however, in the absence of the photon field.

The field equation for Ψ is

$$(D^\mu\Psi)_{|\mu} - iq\, e^{-\Phi}A_\mu\, D^\mu\Psi = 0 \tag{3.35}$$

There is no coupling between Ψ and Φ in the absence of A_μ.

Remarks

Although the gauge theory of Hojman et al. is mathematically consistent, it does not fit into the geometrical framework of fibre bundles. This objection could be dismissed as a pure aesthetic one if it would be clear, how to generalize this concept to the non-abelian case.

And even in the abelian case it is unsatisfactory that the torsion, which in general has 24 independent components, must be subject to the special form (3.21) only in order to make the theory work.

It is fair to say that Hojman et al. have simply introduced a peculiar matter field Φ which interacts nonlinearly with A_μ and Ψ. The interpretation of Φ as a potential of torsion is of secondary importance. The field equations for gravity (associated with $\delta g_{\mu\nu}$)

$$\overset{LC}{G}_{\mu\nu} = 6(\Phi_{,\mu}\,\Phi_{,\nu} - 1/2\, g_{\mu\nu}\,\Phi_{,\alpha}\,\Phi^{,\alpha}) + \\ + 2(F_\mu{}^\alpha F_{\nu\alpha} - 1/4\, g_{\mu\nu}\, F_{\alpha\beta}F^{\alpha\beta}) + 4(D_{(\mu}\Psi^* D_{\nu)}\Psi - 1/2\, g_{\mu\nu} D^\alpha \Psi^* D_\alpha \Psi) \tag{3.36}$$

reduce to the ordinary Einstein equations for vanishing Φ. For nonvanishing Φ the left hand side is unaffected, but on the right hand side stands a modified energy-stress tensor.

Finally we notice that the modified Maxwell equations

$$F^{\mu\nu}{}_{|\nu} + F^{\mu\nu}\,\Phi_{|\nu} = -iq\, e^{-\Phi}\, j^\mu \tag{3.37}$$

$$F_{[\mu\nu|\lambda]} = \Phi_{|[\lambda} A_{\mu,\nu]} - \Phi_{|[\lambda} A_{\nu,\mu]} \tag{3.38}$$

$$\Phi^{|\alpha}{}_{|\alpha} = 1/6\, F^{\mu\nu} F_{\mu\nu} \tag{3.39}$$

are compatible with observation since the source term on the right of (3.39) couples with the strength of the Newtonian gravitation constant, which in units $\hbar = c = 1$ is of the order $G \sim (10^{19}\, m_N)^{-2}$.

The renormalizability is, however, an open question.

IV The Gauge Symmetries of Gravitation

The dynamical field in Einstein's general relativity is the Lorentz metric $g_{\mu\nu}$. In the subsequent chapter we shall give an equivalent formulation by taking the Lorentz frames θ^ν as the dynamical elements. That reformulation is not only useful technically, but allows the interpretation of general relativity as a gauge theory of the translation group T^4, since the Lorentz frames θ^ν can be considered as translational gauge potentials. Such a point of view is, however, rather formal and one may be tempted to look out for more intriguing gauge fields.

Now one might argue that spinor fields have to be incorporated in a Lagrangian field theory of gravitation, and this requires $SL(2,\mathbb{C})$ to be a symmetry group of the Lagrangian. In general relativity one is dealing with the Levi-Cività connection (see section 2.5.). Gauging the group $SL(2,\mathbb{C})$ amounts to replacing the Levi-Cività connection ${}^{LC}\omega^\alpha{}_\beta$ by a metric connection $\omega^\alpha{}_\beta$, or equivalently, to introducing a dynamical contorsion $\tau^\alpha{}_\beta$ (respectively a dynamical torsion $Q^\alpha_{\mu\nu}$), since $\omega^\alpha{}_\beta$ admits a unique decomposition

$$\omega^\alpha{}_\beta = {}^{LC}\omega^\alpha{}_\beta + \tau^\alpha{}_\beta \tag{4.1}$$

The following discussion will show, however, that $SL(2,\mathbb{C})$ should not be a proper internal symmetry group. More explicitly, we shall find that any Lagrangian being locally gauge invariant with respect to $SL(2,\mathbb{C})$ in the strict sense of definition (3.7) provides field equations for $\tau^\alpha{}_\beta$, which allow no unique solution of the general Cauchy problem for $\tau^\alpha{}_\beta$. This criterion rules out, for instance, the YM-type Lagrangian $-1/2\ \Omega^\alpha{}_\beta \wedge {}^*\Omega_\alpha{}^\beta$ for gravitation. In order to save the concept of section 3.1. one might try to describe gravity by a gauge theory involving a compact structure group. But we have no idea, how this could be realized.

In section 4.1. we study the $GL(4,\mathbb{R})$-symmetry of a Lagrangian density. We shall derive a first differential identity. In section 4.2. we compare the concept of local Lorentz invariance to the concept of local gauge invariance in section 3.1. The Cauchy problem for the contorsion is discussed in this context. In section 4.3. we illustrate, in what sense the inhomogeneous Poincaré group may be considered as a gauge group for gravity. In section 4.4. we discuss the diffeomorphism invariance of a Lagrangian and derive a differential identity. A structure theorem, which connects the differential identities with the field equations, is given in section 4.5.

4.1. The GL(4,R)-invariance *

We consider a Lagrangian density L describing the total system of geometry and matter:

$$L = L(\text{geometry},\text{matter})$$

The dynamical elements of geometry are a

$$\begin{aligned} &\text{Lorentz metric} && g_{\mu\nu} \\ &\text{linear frame} && \theta^{\lambda}_{\ } \\ &\text{linear connection} && \omega^{\alpha}_{\ \beta} \end{aligned} \tag{4.2}$$

(At this point $\omega^{\alpha}_{\ \beta}$ is not necessarily metric.)

The dynamics of matter are described by

$$\text{matter fields} \qquad \Psi \tag{4.3}$$

(At this point Ψ is a tensorfield transforming under GL(4,$\mathbb{R}$) according to the tensor representation. Spinorfields are included below.)

We introduce the currents associated with the fields listed under (4.2):

$$\delta_g L = 1/2\ \overset{\circ}{E}^{\mu\nu}\ \delta g_{\mu\nu}\ \eta + \text{exact form} \tag{4.4}$$

$$\delta_\theta L = \delta\theta^{\mu} \wedge E_{\mu} + \text{exact form} \tag{4.5}$$

$$\delta_\omega L = 1/2\ \delta\omega_{\alpha\beta} \wedge S^{\alpha\beta} + \text{exact form} \tag{4.6}$$

$$\overset{\circ}{E}_{\mu\nu} = \overset{\circ}{G}_{\mu\nu} + \overset{\circ}{T}_{\mu\nu} \tag{4.7}$$

$$E_{\mu} = \varepsilon_{\mu} + t_{\mu} \tag{4.8}$$

$$S^{\alpha\beta} = \sigma^{\alpha\beta} + s^{\alpha\beta} \tag{4.9}$$

$$\overset{\circ}{G}_{\mu\nu} : \text{symmetric energy-stress of geometry} \tag{4.10}$$

$$\overset{\circ}{T}_{\mu\nu} : \text{symmetric energy-stress of matter} \tag{4.11}$$

$$\varepsilon_{\mu} : \text{canonical energy-stress of geometry} \tag{4.12}$$

$$t_{\mu} : \text{canonical energy-stress of matter} \tag{4.13}$$

$$\sigma^{\alpha\beta} : \text{spin-density of geometry} \tag{4.14}$$

$$s^{\alpha\beta} : \text{spin-density of matter} \tag{4.15}$$

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* See Ref.{46}

tensor components

$$\varepsilon_\mu = - \mathfrak{g}^\nu{}_\mu \, \eta_\nu \tag{4.16}$$

$$t_\mu = - t^\nu{}_\mu \, \eta_\nu \tag{4.17}$$

$$\sigma^{\alpha\beta} = \Sigma^{\mu\alpha\beta} \, \eta\mu \tag{4.18}$$

$$s^{\alpha\beta} = J^{\mu\alpha\beta} \, \eta_\mu \tag{4.19}$$

Examples are discussed in appendix I.

Independent variations of the fields under (4.2) and (4.3) yield

$$\delta L = 1/2 \, \mathring{E}^{\mu\nu} \, \delta g_{\mu\nu} \, \eta + \delta\theta^\mu \wedge E_\mu + 1/2 \, \delta\omega_{\alpha\beta} \wedge S^{\alpha\beta} + \delta\Psi \wedge L_A + \text{exact form} \tag{4.20}$$

Now we derive a differential identity, which is a consequence of the local GL(4,ℝ)-invariance of the scalar 4-form L. For this purpose let $h(x) = (h^\mu{}_\nu(x))$ be a gl(4,ℝ)-valued function, which vanishes on M-D for some compact region D with regular boundary. For arbitrary $s \in \mathbb{R}$ we put $g_s(x) = \exp(s.h(x))$. Then one has

$$L\{g_{\mu\nu}(x);\theta^\mu(x);\omega^\alpha{}_\beta(x);\Psi(x);D\Psi(x)\} =$$

$$L\{g_{-s}(x)^\lambda_\mu \, g_{-s}(x)^\rho_\nu \, g_{\lambda\rho}(x); g_s(x)^\mu_\nu \theta^\nu(x); g_s(x)^\alpha_\rho g_{-s}(x)^\lambda_\beta \omega^\rho_\lambda(x) - dg_s(x)^\alpha_\lambda \, g_{-s}(x)^\lambda_\beta;$$
$$S(g_s(x))\Psi(x); S(g_s(x)) \, D\Psi(x) \} + \text{exact differential} \tag{4.21}$$

We differentiate (4.21) with respect to s at s=0 and observe equation (4.20) ; then we find

$$0 = - 1/2 \, \mathring{E}^{\mu\nu}(h_{\mu\nu} + h_{\nu\mu}) + h^\mu{}_\nu \theta^\nu \wedge E_\mu - 1/2 \, Dh_{\alpha\beta} \wedge S^{\alpha\beta} +$$
$$+ S_*(h)\Psi \wedge L_A + \text{exact form} =$$

$$= - h_{\mu\nu}(\mathring{E}^{\mu\nu} \eta - \theta^\nu \wedge E^\mu - 1/2 \, DS^{\mu\nu}) + S_*(h)\Psi \wedge L_A + \text{exact form} \tag{4.22}$$

Integrating (4.22) over D we conclude , if the equations $L_A = 0$ for the matter fields are satisfied, the following identity

$$\mathring{E}^{\mu\nu} \, \eta = \theta^\nu \wedge E^\mu + 1/2 \, DS^{\mu\nu} \quad (\text{modulo } L_A = 0) \tag{4.23}$$

field equations

Independent variations of $g_{\mu\nu}$ and $\omega^\alpha{}_\beta$, respectively of θ^μ and $\omega^\alpha{}_\beta$, yield the following formal field equations for the geometry:

$$\overset{\ell}{E}{}^{\mu\nu} = 0 \qquad (L_A = 0) \tag{4.24}$$

$$S^{\alpha\beta} = 0 \tag{4.25}$$

respectively

$$E^{\mu} = 0 \qquad (L_A = 0) \tag{4.26}$$

$$S^{\alpha\beta} = 0 \tag{4.27}$$

Thanks to the identity (4.23), these two systems are equivalent.

4.2. Local Lorentz invariance and the Cauchy problem

We consider gauge theories on (M,g) having $L_+^{\uparrow}$, respectively SL(2,$\mathbb{C}$) as structure group. The fibre bundle then is a Lorentz frame bundle $(\mathcal{Q}^{\Lambda},M)$ respectively an associated covering spinor bundle $(\mathcal{Q}^{SL},M)$.

The dynamical elements are a <u>metric</u> connection $\omega^{\alpha}{}_{\beta}$, the matter fields Ψ and in addition the Lorentz frames θ^{ν}. The total Lagrangian is a functional of $\Omega^{\alpha}{}_{\beta},\theta^{\mu},\Theta^{\mu} = D\theta^{\mu},\Psi$ and $D\Psi$, i.e. it is a 4-form

$$L\{\Omega^{\alpha}{}_{\beta},\theta^{\mu},\Theta^{\mu},\Psi,D\Psi\} \tag{4.28}$$

We first give a definition, which is the strict analogon of (3.7):

<u>Definition:</u> L satisfies the <u>strong local Lorentz invariance</u>, if for <u>fixed</u> $\theta^{\mu},\theta^{\mu}\wedge\theta^{\nu}$ etc., and <u>fixed</u> $\eta^{\mu},\eta^{\mu\nu}$ etc., L is invariant under the simultaneous transformations

$$\Psi \longrightarrow \Psi' = S(\Lambda)\Psi$$

$$\omega^{\alpha}{}_{\beta} \longrightarrow \omega'^{\alpha}{}_{\beta} = \Lambda^{\alpha}{}_{\lambda}\omega^{\lambda}{}_{\sigma}\bar{\Lambda}^{-1\sigma}{}_{\beta} - d\Lambda^{\alpha}{}_{\lambda}.\bar{\Lambda}^{-1\lambda}{}_{\beta}$$

for arbitrary locally defined functions $\Lambda(x)\in L_+^{\uparrow}$. (4.29)

An example is the sum of the Lagrangian of a massive scalar field and of the YM-type Lagrangian for gravitation

$$L = -1/2\ \Omega^{\alpha}{}_{\beta}\wedge{}^*\Omega_{\alpha}{}^{\beta} + \kappa/2\ (d\Phi\wedge{}^*d\Phi + m^2\Phi^2\eta) \tag{4.30}$$

In (4.30) the frames and their duals do not appear explicitly and the volume η is unchanged under Lorentz rotations of the frames.

By contrast, the Dirac Lagrangian

$$L = i\,\bar{\Psi}\gamma^{\mu}D\Psi\wedge\eta_{\mu} - m\,\bar{\Psi}\Psi\,\eta \tag{4.31}$$

does not have property (4.29). A transformation of the spin-indices of Ψ induces a transformation of the vector-spin index μ of γ^{μ} and this latter index is contracted with η_{μ}.

A weakened version of the definition (4.29) is the following

Definition : L satisfies the weak local Lorentz invariance, if L is invariant under the simultaneous transformations

$$\begin{aligned}
\Psi &\longrightarrow \Psi' &&= S(\Lambda)\Psi \\
\omega^{\alpha}{}_{\beta} &\longrightarrow \omega'^{\alpha}{}_{\beta} &&= \Lambda^{\alpha}{}_{\lambda}\omega^{\lambda}{}_{\sigma}\bar{\Lambda}^{1\sigma}{}_{\beta} - d\Lambda^{\alpha}{}_{\lambda}.\bar{\Lambda}^{1\lambda}{}_{\beta} \\
\theta^{\mu} &\longrightarrow \theta'^{\mu} &&= \Lambda^{\mu}{}_{\nu}\theta^{\nu} \\
\theta^{\mu}\wedge\theta^{\nu} &\longrightarrow \theta'^{\mu}\wedge\theta'^{\nu} &&= \Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta}\,\theta^{\alpha}\wedge\theta^{\beta}
\end{aligned}$$

etc.

for arbitrary locally defined functions $\Lambda(x)\in L^{\uparrow}_{+}$. (4.32)

It is easy to see that (4.31) is invariant in this weaker sense.

The following considerations on the level of infinitesimal local Lorentz transformations are helpful to emphasize the difference between (4.29) and (4.32). For this purpose let L fulfil property (4.32). With the equations $L_A = 0$ for the matter fields we find

$$\delta L = \delta\theta^{\mu}\wedge E_{\mu} + 1/2\;\delta\omega^{\alpha}{}_{\beta}\wedge S_{\alpha}{}^{\beta} = 0 \tag{4.33}$$

If L has property (4.29) we get no contribution from the local (infinitesimal) Lorentz rotations of the frames, i.e.

$$\delta\theta^{\mu}\wedge E_{\mu} = 0 \tag{4.34}$$

and consequently

$$\delta\omega^{\alpha\beta}\wedge S_{\alpha\beta} = 0 \tag{4.35}$$

Conversely, if (4.34) and (4.35) hold, then L has property (4.29).

Integration of (4.33) over compact regions yields the following identity:

$$DS^{\alpha\beta} = \theta^{\alpha}\wedge E^{\beta} - \theta^{\beta}\wedge E^{\alpha} = (B^{\alpha\beta})^{\mu}{}_{\nu}(t^{\nu}{}_{\mu} + \sigma\gamma^{\nu}{}_{\mu})\eta \tag{4.36}$$

If L satisfies the strong local Lorentz invariance then each side of (4.36) is separately equal to zero, i.e.

$$DS^{\alpha\beta} = 0 \quad \text{and} \quad t^{\mu\nu} + \mathfrak{g}^{\mu\nu} = t^{\nu\mu} + \mathfrak{g}^{\nu\mu} \tag{4.37}$$

Now we come to the crucial point of this discussion: If L satisfies the strong local Lorentz invariance, then we know from (4.35) that

$$\delta_\omega L = 0 \tag{4.38}$$

for all variations $\delta\omega^\alpha{}_\beta = -Dh^\alpha{}_\beta$; $h_{\alpha\beta}(x) = - h_{\beta\alpha}(x)$.

At this point we make use of the unique decomposition (4.1) of ω into the Levi-Civitá part $\overset{LC}{\omega}$ plus the contorsion 1-form τ. With (4.1), property (4.38) says that L is invariant under the substitutions

$$\overset{LC}{\omega}{}^\alpha{}_\beta + \tau^\alpha{}_\beta \longrightarrow \overset{LC}{\omega}{}^\alpha{}_\beta + (\tau^\alpha{}_\beta - Dh^\alpha{}_\beta) \tag{4.39}$$

Since $Dh^\alpha{}_\beta$ is a tensor 1-form of the same type as $\tau^\alpha{}_\beta$, (4.38) amounts to the statement that $\tau^\alpha{}_\beta$ can be replaced by $\tilde{\tau}^\alpha{}_\beta = \tau^\alpha{}_\beta - Dh^\alpha{}_\beta$ without affecting L ! But if this is true, the system of differential equations resulting from variations with respect to θ^μ and $\omega^\alpha{}_\beta$ cannot be complete!

This proves our statement at the beginning of this chapter that $L_+^\uparrow$ respectively $SL(2,\mathbb{C})$ should not be a pure internal symmetry group.

4.3. The Poincaré group as a gauge group *

Although mathematically correct, it is mere semantics to say that the pair $(\omega^\alpha{}_\beta, \theta^\mu)$ is a connection associated with the inhomogeneous Lorentz group, i.e. with the Poincaré group. The operation of the the translation group T^4 on any fields is, as dictated by the theory of principal fibre bundles (see Ref.{27}), the trivial one. Nevertheless, it is at least intriguing to consider gravitation theories with dynamical torsion as gauge theories of the Poincaré group.

*See also appendix II.

Independent variations of $\omega^{\alpha}{}_{\beta}$ and θ^{μ} yield

$$\delta L = 1/2\ \omega_{\alpha\beta}\wedge S^{\alpha\beta} + \delta\theta^{\mu}\wedge E_{\mu} \tag{4.40}$$

and the respective field equations are

$$E_{\mu} = 0 \tag{4.41}$$

$$S_{\alpha\beta} = 0 \tag{4.42}$$

One may object that the only dynamically free part of $\omega^{\alpha}{}_{\beta}$ is $\tau^{\alpha}{}_{\beta}$ and that one has to vary with respect to $\tau^{\alpha}{}_{\beta}$, i.e.

$$\delta L = 1/2\ \delta\tau_{\alpha\beta}\wedge S^{\alpha\beta} + \delta\theta^{\mu}\wedge E_{\mu} \tag{4.43}$$

But this approach leads , obviously, to the same field equations (4.41) and (4.42).

What happens if one takes the frame induced variations $\delta\overset{LC}{\omega}{}^{\alpha}{}_{\beta}\{\delta\theta^{\mu}\}$ into account? (See formula (2.35)). The modification of (4.43) would be

$$\Delta L = 1/2\ \delta\tau_{\alpha\beta}\wedge S^{\alpha\beta} + \delta\theta^{\mu}\wedge E_{\mu} + 1/2\ \delta\overset{LC}{\omega}{}_{\alpha\beta}\wedge S^{\alpha\beta} \tag{4.44}$$

[At this point we assume that all curvatures belong to metric connections with torsion. We do not admit Lagrangians of the type $-1/2\ \overset{LC}{\Omega}{}^{\alpha}{}_{\beta}\wedge{}^{*}\overset{LC}{\Omega}{}_{\alpha}{}^{\beta}$.]

But the field equation associated with $\delta\tau_{\alpha\beta}$ is $S_{\alpha\beta} = 0$ and thus $1/2\ \delta\overset{LC}{\omega}{}_{\alpha\beta}\wedge S^{\alpha\beta}$ drops out in (4.44), i.e. the field equation associated with $\delta\theta^{\mu}$ is again $E_{\mu} = 0$.

4.4. The diffeomorphism group Diff(M)

When describing space time by Lorentz manifolds , two models (M,g) and (M',g') are taken to be equivalent, if they are isometric, i.e. if there is a diffeomorphism

$$\Phi: M \longrightarrow M'$$

which carries the metric g into the metric g' , i.e. if

$$\Phi^{*}g' = g$$

In particular we consider the group Diff(M) of all diffeomorphisms of a Lorentz manifold (M,g) upon itself. The local dynamical elements of a Poincaré gauge theory are a metric connection $\omega^{\alpha}{}_{\beta}$, a Lorentz frame θ^{μ} and matter fields $\Psi = \Psi^{\alpha}E_{\alpha}$. Clearly, the

transformed fields

$$\Phi^*\theta^\mu \quad , \quad \Phi^*\omega^\alpha{}_\beta \quad , \quad \text{and} \quad \Phi^*\Psi := (\Phi^*\Psi^\alpha)\, E_\alpha$$

describe on (M,Φ^*g) the same physical situation as the fields θ^μ, $\omega^\alpha{}_\beta$ and Ψ do on (M,g) , if Φ is an arbitrary element of Diff(M). This is guaranted, if the Lagrangian is an invariant functional of the dynamical fields, i.e. if

$$\Phi^*\{L(\theta^\mu,\omega^\alpha{}_\beta,\Psi)\} = L(\Phi^*\theta^\mu,\Phi^*\omega^\alpha{}_\beta,\Phi^*\Psi) \tag{4.45}$$

for all $\Phi\varepsilon$Diff(M).

The following identity is a consequence of the Diff(M)-invariance (4.45) of L :

$$DE_\mu = Q_\mu^\nu \wedge E_\nu + 1/2\, R^{\alpha\beta}{}_\mu \wedge S_{\alpha\beta} \tag{4.46}$$

where

$$Q_\mu^\nu = Q_{\mu\lambda}^\nu\, \theta^\lambda \quad \text{and} \quad R^{\alpha\beta}{}_\mu = R^{\alpha\beta}{}_{\mu\nu}\theta^\nu .$$

In private discussions with his students Prof.N.Straumann has shown the following straightforward proof of (4.46) :

Let D be a compact region with regular boundary and X a vector-field with $\mathrm{supp}(X)\subset D$. Then the flow $\Phi(x,s)$ associated with X is complete. For any $s\varepsilon\mathbb{R}$ one has $\Phi_s.D = D$. Obviously, Φ_s may be regarded as a one-parameter subgroup of Diff(M), which leaves the points outside of D fixed.

From (4.45) we conclude

$$\frac{d}{ds}\Big|_0 \int_D L(\Phi_s^*\theta^\mu,\Phi_s^*\omega^\alpha{}_\beta,\Phi_s^*\Psi) = \frac{d}{ds}\Big|_0 \int_D \Phi_s^*\{L(\theta^\mu,\omega^\alpha{}_\beta,\Psi)\} =$$

$$= \int_D L_X L(\theta^\mu,\omega^\alpha{}_\beta,\Psi) = \int_{\partial D} i_X L(\theta^\mu,\omega^\alpha{}_\beta,\Psi) = 0$$

In addition we notice that

$$\frac{d}{ds}\Big|_0 L(\Phi_s^*\theta^\mu,\Phi_s^*\omega^\alpha{}_\beta,\Phi_s^*\Psi) = L_X\theta^\mu\wedge E_\mu + 1/2\, L_X\omega^\alpha{}_\beta \wedge S_\alpha{}^\beta + L_X\Psi\wedge L_A$$

With the equations $L_A = 0$ for the matter fields we are left with

$$\int_D L_X\theta^\mu\wedge E_\mu + 1/2\, L_X\omega_{\alpha\beta}\wedge S^{\alpha\beta} = 0 \tag{4.47}$$

With the Cartan formula

$$L_X = i_X.d + d.i_X \tag{4.48}$$

and with (2.22) we have

$$L_X\theta^\mu\wedge E_\mu = i_X(\Theta^\mu - \omega^\mu_\nu\wedge\theta^\nu)\wedge E_\mu - i_X\theta^\mu\wedge dE_\mu$$

and hence

$$\int_D L_X\theta^\mu\wedge E_\mu = \int_D i_X(\Theta^\mu - \omega^\mu{}_\nu\wedge\theta^\nu)\wedge E_\mu - \int_D i_X\theta^\mu\wedge(DE_\mu + \omega^\nu_\mu\wedge E_\nu) =$$

$$= \int_D -i_X\theta^\mu\wedge DE_\mu + i_X\Theta^\mu\wedge E_\mu - i_X\omega^\mu{}_\nu\wedge\theta^\nu\wedge E_\mu \qquad (4.49)$$

Similarly we have

$$L_X\omega^{\alpha\beta} = d.i_X\omega^{\alpha\beta} + i_X\Omega^{\alpha\beta} - i_X\omega^\alpha{}_\nu.\omega^{\nu\beta} + \omega^\alpha{}_\nu.i_X\omega^{\nu\beta}$$

Using this we rewrite the second term in (4.47)

$$1/2\ L_X\omega_{\alpha\beta}\wedge S^{\alpha\beta} = -1/2\ DS_{\alpha\beta}.i_X\omega^{\alpha\beta} - 1/2\ S_{\alpha\beta}\wedge i_X\Omega^{\alpha\beta}$$

or

$$1/2\int_D L_X\omega_{\alpha\beta}\wedge S^{\alpha\beta} = -1/2\int_D S_{\alpha\beta}\wedge i_X\Omega^{\alpha\beta} + DS_{\alpha\beta}.i_X\omega^{\alpha\beta} \qquad (4.50)$$

Taking (4.47), (4.49) and (4.50) together yields

$$\int_D -i_X\theta^\mu\wedge DE_\mu + i_X\Theta^\mu\wedge E_\mu - i_X\omega^\mu_\nu\wedge\theta^\nu\wedge E_\mu -$$

$$- 1/2\ S_{\alpha\beta}\wedge i_X\Omega^{\alpha\beta} - 1/2\ DS_{\alpha\beta}\wedge i_X\omega^{\alpha\beta} = 0 \qquad (4.51)$$

Now we bring the <u>local Lorentz invariance</u> into play by rewriting $-i_X\omega^\mu{}_\nu\wedge\theta^\nu\wedge E_\mu$ with the help of (4.36)

$-i_X\omega^\mu{}_\nu\wedge\theta^\nu\wedge E_\mu = -1/2\ i_X\omega^{\mu\nu}(\theta_\nu\wedge E_\mu - \theta_\mu\wedge E_\nu) =$

$= -1/2\ i_X\omega^{\mu\nu}\ DS_{\nu\mu} = 1/2\ DS_{\mu\nu}.i_X\omega^{\mu\nu}$. This last term then drops out in (4.51) :

$$\int_D -i_X\theta^\mu\wedge DE_\mu + i_X\Theta^\mu\wedge E_\mu - 1/2\ S_{\alpha\beta}\wedge i_X\Omega^{\alpha\beta} = 0 \qquad (4.52)$$

Since $i_X\Theta^\mu = Q^\mu{}_\lambda\ i_X\theta^\lambda$ and similarly $i_X\Omega^{\alpha\beta} = R^{\alpha\beta}{}_\lambda\ i_X\theta^\lambda$ one finally has

$$\int_D \{Q^\mu{}_\lambda\wedge E_\mu - DE_\mu - 1/2S_{\alpha\beta}\wedge R^{\alpha\beta}{}_\lambda\}.i_X\theta^\lambda = 0 \qquad (4.53)$$

Since this holds for all such vector fields X the identity (4.46) is proven.

<u>Remark :</u> On Minkowski space (4.46) reduces to $t^{\nu\mu}{}_{,\nu} = 0$ and (4.36) becomes $t^{\beta\alpha} - t^{\alpha\beta} = J^{\nu\alpha\beta}{}_{,\nu}$. But these two identities imply that the angular momentum density

$$M^{\mu\alpha\beta} = t^{\mu\alpha}\ x^\beta - t^{\mu\beta}\ x^\alpha - J^{\mu\alpha\beta} \qquad (4.54)$$

is conserved, i.e. $M^{\mu\alpha\beta}{}_{,\beta} = 0$.

This illustrates the feature that conservation laws on Minkowski space generalize to differential identities on a curved Lorentz manifold.

4.5. Field equations and the differential identities

We collect (4.36), (4.46) and the field equations in the following list

Local Lorentz invariance

$$D\sigma^{\alpha\beta} = \theta^{\alpha}\wedge\varepsilon^{\beta} - \theta^{\beta}\wedge\varepsilon^{\alpha} \tag{4.55}$$

$$Ds^{\alpha\beta} = \theta^{\alpha}\wedge t^{\beta} - \theta^{\beta}\wedge t^{\alpha} \qquad (L_A = 0) \tag{4.56}$$

Diff(M)-invariance

$$D\varepsilon_{\mu} = Q^{\nu}{}_{\mu}\wedge\varepsilon_{\nu} + 1/2\ R^{\alpha\beta}{}_{\mu}\wedge\sigma_{\alpha\beta} \tag{4.57}$$

$$Dt_{\mu} = Q^{\nu}{}_{\mu}\wedge t_{\nu} + 1/2\ R^{\alpha\beta}{}_{\mu}\wedge s_{\alpha\beta} \qquad (L_A = 0) \tag{4.58}$$

Field equations

$$\varepsilon_{\mu} = -\,t_{\mu} \tag{4.59}$$

$$\sigma_{\alpha\beta} = -\,s_{\alpha\beta} \tag{4.60}$$

This list implies the following theorem (first formulated by A.Trautman{46} in the special case of the Einstein-Cartan theory) :

> The equations resulting by covariant exterior derivation of (4.59) and (4.60) are algebraic consequences of the above listed identities and of the field equations themselves. (4.61)

This theorem shows that, in order to get a consistent theory , both equations (4.59) and (4.60) are needed simultaneously.

V The Levi-Cività Formulation of Gravitation

In general relativity the dynamical element of gravity is the Lorentz metric , and the metric connection is always the Levi-Cività connection. Both sides of the Einstein field equations, the Einstein tensor and the energy-stress tensor, are symmetric and (Levi-Cività) conserved. We say that such a Lagrangian field theory of gravitation is of the Levi-Cività type.

In section 5.1. we derive a differential identity, which connects the canonical currents (4.11),(4.13), and (4.15).In section 5.2. we express the symmetric and (Levi-Cività) conserved energy-stress tensor in terms of (4.13) and (4.15) respectively in terms of (4.11) and (4.15). A complete table of the Levi-Cività identities is given in section 5.3. In section 5.4. we show, in what sense Lagrangian field theories of the Levi-Cività type can be understood as gauge theories of the translational group T^4 . In section 5.5. we reformulate the gauge theories of gravitation, described in chapter IV , as Lagrangian field theories of the Levi-Cività type. Torsion then appears as a matter field. This anti-gauge-theoretic point of view illustrates the special nature of gravity.

5.1. The relation between $\overset{\circ}{T}{}^{\mu\nu}$, $t^{\mu\nu}$ and $J^{\lambda\mu\nu}$

We consider the symmetric energy-stress tensor $\overset{\circ}{T}{}^{\mu\nu}$, the canonical energy-stress tensor t_μ and the spin-density $s^{\mu\nu}$ associated with some matter Lagrangian $L(\Psi,D\Psi)$. The following discussion holds also for Lagrangians including gravitational fields.

The GL(4,R)-invariance (4.23), respectively the local Lorentz invariance (4.36), reads as

$$\overset{\circ}{T}{}^{\mu\nu}\,\eta = \theta^\nu \wedge t^\mu + 1/2\, Ds^{\mu\nu} \tag{5.1}$$

respectively as

$$Ds^{\mu\nu} = \theta^\mu \wedge t^\nu - \theta^\nu \wedge t^\mu \tag{5.2}$$

Combining these two equations yields

$$\overset{\circ}{T}{}^{\mu\nu}\,\eta = 1/2(\theta^\mu \wedge t^\nu + \theta^\nu \wedge t^\mu) \tag{5.3}$$

respectively , with (2.9) and (4.17),

$$\overset{\circ}{T}{}^{\mu\nu} = -1/2 \ (t^{\mu\nu} + t^{\nu\mu}) \tag{5.4}$$

In components equation (5.2) says that

$$J^{\lambda\mu\nu}{}_{;\lambda} + J^{\lambda\mu\nu} \, Q^{\alpha}_{\lambda\alpha} = - t^{\mu\nu} + t^{\nu\mu} \tag{5.5}$$

or

$$t^{\nu\mu} = t^{\mu\nu} + J^{\lambda\mu\nu}{}_{;\lambda} + J^{\lambda\mu\nu} \, Q^{\alpha}_{\lambda\alpha} \tag{5.6}$$

We combine this with (5.4) and find

$$\overset{\circ}{T}{}^{\mu\nu} = - t^{\nu\mu} + 1/2 \, J^{\lambda\mu\nu}{}_{;\lambda} + 1/2 \, J^{\lambda\mu\nu} \, Q^{\alpha}_{\lambda\alpha} \tag{5.7}$$

For the Levi-Cività connection this reads as

$$\overset{\circ}{T}{}^{\mu\nu} = - t^{\nu\mu} + 1/2 \, J^{\lambda\mu\nu}{}_{|\lambda} \tag{5.8}$$

By (4.46) one has on Minkowski space $t^{\nu\mu}{}_{,\nu} = 0$. Taking (5.8) into account, we gain an identity for the divergence of $\overset{\circ}{T}{}^{\mu\nu}$ on Minkowski space :

$$\overset{\circ}{T}{}^{\mu\nu}{}_{,\nu} = 1/2 \, J^{\lambda\mu\nu}{}_{,\lambda\nu} \tag{5.9}$$

But the right hand side of (5.9) does not vanish in the sense of a differential identity. This shows that $\overset{\circ}{T}{}^{\mu\nu}$ cannot be the usual symmetric energy-stress tensor of general relativity.

5.2. The conserved and symmetric energy-stress tensor $T^{\mu\nu}$

As in the previous section we consider some matter Lagrangian $L = L(\Psi, \overset{LC}{D}\Psi)$ but the connection involved here is exclusively the Levi-Cività connection. The dynamical elements of gravitation are the Lorentz frames θ^{ν} or, alternatively, the metric $g_{\mu\nu}$. The following discussion holds also for gravitational Lagrangians.

We define a new energy-stress tensor $T^{\mu} = -T^{\nu\mu} \, \eta_{\nu}$ by taking the frame induced variations of $\overset{LC}{\omega}{}_{\alpha\beta}$ into account

$$\Delta_{\theta} L = \Delta\theta_{\mu} \wedge T^{\mu} = \delta\theta_{\mu} \wedge t^{\mu} + 1/2 \, \delta\overset{LC}{\omega}{}_{\mu\nu} \wedge s^{\mu\nu} \tag{5.10}$$

From formula (2.35) we conclude

$$1/2 \, \delta\overset{LC}{\omega}{}_{\mu\nu} \wedge s^{\mu\nu} = f^{\nu\lambda\mu}{}_{|\lambda} \, \delta\theta_{\mu} \wedge \eta_{\nu} \quad , \text{ where}$$

$$f^{\mu\nu\lambda} = 1/2 \, (J^{\mu\nu\lambda} + J^{\nu\lambda\mu} - J^{\lambda\mu\nu}) \tag{5.11}$$

By (5.10) this yields

$$T^{\mu} = t^{\mu} + f^{\nu\lambda\mu}{}_{|\lambda}\,\eta_{\nu} \tag{5.12}$$

As a consequence of (5.5) one has the symmetry of $T^{\mu\nu}$

$$T^{\mu\nu} - T^{\nu\mu} = t^{\mu\nu} - t^{\nu\mu} + (f^{\nu\lambda\mu} - f^{\mu\lambda\nu})_{|\lambda} = t^{\mu\nu} - t^{\nu\mu} + J^{\lambda\mu\nu}{}_{|\lambda} = 0$$

As in section 4.4. we evaluate the Diff(M)-invariance of L. Equation (4.47) then reads as

$$0 = \int_D L_X\theta^{\mu}\wedge t_{\mu} + 1/2\ \delta\overset{LC}{\omega}{}_{\alpha\beta}\{L_X\theta^{\mu}\}\wedge s^{\alpha\beta} = \int_D L_X\theta^{\mu}\wedge T_{\mu} =$$

$$= -\int_D i_X\theta^{\mu}\wedge \overset{LC}{D}T_{\mu} - \int_D i_X\overset{LC}{\omega}{}_{\alpha\beta}\wedge\theta^{\beta}\wedge T^{\alpha}$$

By the symmetry of $T^{\alpha\beta}$ the last integral vanishes separately and hence we conclude

$$\overset{LC}{D}T_{\mu} = 0 \quad \text{respectively} \quad T^{\mu\nu}{}_{|\nu} = 0 \tag{5.13}$$

From (5.12) one has

$$T^{\mu\nu} = t^{\mu\nu} - f^{\mu\lambda\nu}{}_{|\lambda} \tag{5.14}$$

Taking the metric induced variations

$$\delta\overset{LC}{\omega}{}_{\alpha\beta}\{\delta g_{\mu\nu}\} = 1/2\ (\delta g_{\lambda\alpha|\beta} + \delta g_{\alpha\beta|\lambda} - \delta g_{\beta\lambda|\alpha})\theta^{\lambda} \tag{5.15}$$

into account, the components (5.14) result also from variations of L with respect to $g_{\mu\nu}$, i.e.

$$\Delta_g L = 1/2\ \overset{\circ}{T}{}^{\mu\nu}\ \delta g_{\mu\nu}\ \eta + 1/2\ \delta\overset{LC}{\omega}{}_{\alpha\beta}\{\delta g_{\mu\nu}\}\wedge s^{\alpha\beta} =$$

$$= -1/2\ T^{\mu\nu}\ \Delta g_{\mu\nu}.\eta \tag{5.16}$$

For the proof of (5.16) one needs the identity

$$T^{\mu\nu} = -\overset{\circ}{T}{}^{\mu\nu} - 1/2\ (J^{\mu\lambda\nu} + J^{\nu\lambda\mu})_{|\lambda} \tag{5.17}$$

5.3. Table of the Levi-Cività identities

Let $L = L_g + L_m$ be a Lagrangian for the coupled system of gravitation and matter. We put

$$\Delta_{\theta}L_g = \Delta\theta^{\mu}\wedge G_{\mu}\ ;\ G_{\mu} = -G^{\nu}{}_{\mu}\,\eta_{\nu} \tag{5.18}$$

We apply the above results to the quantities listed in (4.10) ---------(4.19) :

$$\overset{\circ}{G}{}^{\mu\nu}\,\eta = 1/2\ (\theta^{\mu}\wedge\varepsilon^{\nu} + \theta^{\nu}\wedge\varepsilon^{\mu}) = -1/2\ (\sigma^{\mu\nu} + \sigma^{\nu\mu})\cdot\eta \tag{5.19}$$

$$\overset{\circ}{T}{}^{\mu\nu}\,\eta = 1/2\ (\theta^{\mu}\wedge t^{\nu} + \theta^{\nu}\wedge t^{\mu}) = -1/2\ (t^{\mu\nu} + t^{\nu\mu})\cdot\eta \tag{5.20}$$

$$G^{\mu} = \varepsilon^{\mu} + 1/2\ (\Sigma^{\nu\lambda\mu} + \Sigma^{\lambda\mu\nu} - \Sigma^{\mu\nu\lambda})_{|\lambda}\ \eta_{\nu} \tag{5.21}$$

$$T^{\mu} = t^{\mu} + 1/2\ (J^{\nu\lambda\mu} + J^{\lambda\mu\nu} - J^{\mu\nu\lambda})_{|\lambda}\ \eta_{\nu} \tag{5.22}$$

$$G^{\mu\nu} = \sigma^{\mu\nu} - 1/2(\Sigma^{\mu\lambda\nu} + \Sigma^{\lambda\nu\mu} - \Sigma^{\nu\mu\lambda})_{|\lambda} \tag{5.23}$$

$$T^{\mu\nu} = t^{\mu\nu} - 1/2(J^{\mu\lambda\nu} + J^{\lambda\nu\mu} - J^{\nu\mu\lambda})_{|\lambda} \tag{5.24}$$

$$G^{\mu\nu} = -\overset{\circ}{G}{}^{\mu\nu} - 1/2\ (\Sigma^{\mu\lambda\nu} + \Sigma^{\nu\lambda\mu})_{|\lambda} \tag{5.25}$$

$$T^{\mu\nu} = -\overset{\circ}{T}{}^{\mu\nu} - 1/2\ (J^{\mu\lambda\nu} + J^{\nu\lambda\mu})_{|\lambda} \tag{5.26}$$

$$G^{\mu\nu}{}_{|\nu} = 0 \tag{5.27}$$

$$T^{\mu\nu}{}_{|\nu} = 0 \tag{5.28}$$

$$\overset{\circ}{G}{}^{\mu\nu} = -\sigma^{\nu\mu} + 1/2\ \Sigma^{\lambda\mu\nu}{}_{|\lambda} \tag{5.29}$$

$$\overset{\circ}{T}{}^{\mu\nu} = -t^{\nu\mu} + 1/2\ J^{\lambda\mu\nu}{}_{|\lambda} \tag{5.30}$$

5.4. Field equations

Let $L = L_g + L_m$ be as before. The equations resulting from variations $\Delta_\theta L$ with respect to the Lorentz frames θ^μ are

$$G_{\mu} = -T_{\mu} \tag{5.31}$$

or equivalently

$$G_{\mu\nu} = -T_{\mu\nu} \tag{5.32}$$

These equations can be obtained by variations $\Delta_g L$ with respect to the metric $g_{\mu\nu}$. By (5.27) and (5.28) both sides of (5.32) are conserved separately.

A special example is <u>general relativity</u> with

$$L_g = 1/2\ \eta_{\mu\nu}\wedge\overset{LC}{\Omega}{}^{\mu\nu} \tag{5.33}$$

Since the dynamical Lorentz frames θ^ν may be considered as translational gauge potentials it is correct, although rather formal, to regard general relativity as a gauge theory of the translation group T^4 . We shall come back to this point of view in chapter VI.

5.5. The Levi-Cività formalism including dynamical torsion

Any total Lagrangian of the type

$$L_{tot} = L_g(\text{geometry with torsion}) + L_m(\Psi, D\Psi) \tag{5.34}$$

admits a unique decomposition into a pure geometrical part L'_g containing no torsion plus a generalized matter Lagrangian $L'_m(\Psi, \overset{LC}{D}\Psi, \tau^\alpha{}_\beta, \overset{LC}{D}\tau^\alpha{}_\beta)$, which collects the pure matter terms and all the terms involving torsion

$$L_{tot} = L_g + L_m \equiv L'_g + L'_m \tag{5.35}$$

{All curvatures and covariant derivatives in (5.34) are understood with respect to a metric connection with torsion. See the remark (in brackets) in section 4.3. For instance the exact form $1/2\ \eta_{\alpha\beta} \wedge \overset{LC}{D}\tau^{\alpha\beta} = 1/2\ d(\eta_{\alpha\beta} \wedge \tau^{\alpha\beta})$ contained in the gravitational Lagrangian $1/2\ \eta_{\alpha\beta} \wedge \Omega^{\alpha\beta}$ should not be dropped.}

For instance

$$L_{tot} = -1/2\ \Omega^\alpha{}_\beta \wedge {}^*\Omega_\alpha{}^\beta + L_m \tag{5.36}$$

can be rewritten as

$$L_{tot} = -1/2\ \overset{LC}{\Omega}{}^\alpha{}_\beta \wedge {}^*\overset{LC}{\Omega}{}_\alpha{}^\beta + L'_m \tag{5.37}$$

where

$$L'_m = L_m - 1/2\ (\overset{LC}{D}\tau^\alpha_\beta + \tau^\alpha{}_\lambda \wedge \tau^\lambda{}_\beta) \wedge {}^*\Omega_\alpha{}^\beta - \\ - 1/2\ \Omega^\alpha_\beta \wedge {}^*(\overset{LC}{D}\tau_\alpha{}^\beta + \tau_{\alpha\lambda} \wedge \tau^{\lambda\beta}) \tag{5.38}$$

In the following discussion we consider the contorsion $\tau^\alpha{}_\beta$ as a matter field. In this sense we can say that (5.38) collects the pure matter terms and all the coupling terms of the matter fields with geometry. Of course, there are Lagrangians , which are not of the type (5.34). But in such cases L_{tot} must be treated as a single piece. We give the various currents associated with (5.35) :

$$\begin{aligned}
\delta_\theta L_g &= \delta\theta^\mu \wedge \varepsilon_\mu & \qquad \delta_\theta L'_g &= \delta\theta^\mu \wedge \varepsilon'_\mu \\
\delta_\theta L_m &= \delta\theta^\mu \wedge t_\mu & \qquad \delta_\theta L'_m &= \delta\theta^\mu \wedge t'_\mu \\
\Delta_\theta L_g &= \Delta\theta^\mu \wedge G_\mu & \qquad \Delta_\theta L'_g &= \Delta\theta^\mu \wedge G'_\mu \\
\Delta_\theta L_m &= \Delta\theta^\mu \wedge T_\mu & \qquad \Delta_\theta L'_m &= \Delta\theta^\mu \wedge T'_\mu \\
\delta_\tau L_g &= 1/2\ \delta\tau_{\alpha\beta} \wedge \sigma^{\alpha\beta} & \qquad \delta_\tau L'_g &= 0 \\
\delta_\tau L_m &= 1/2\ \delta\tau_{\alpha\beta} \wedge s^{\alpha\beta} & \qquad \delta_\tau L'_m &= 1/2\ \delta\tau_{\alpha\beta} \wedge (s^{\alpha\beta} + \sigma^{\alpha\beta})
\end{aligned} \tag{5.39}$$

The last two equations on the right of (5.39) are a consequence of the form of the decomposition $L'_g + L'_m$. Clearly one has

$$\varepsilon_\mu + t_\mu = \varepsilon'_\mu + t'_\mu \tag{5.40}$$

$$G_\mu + T_\mu = G'_\mu + T'_\mu \tag{5.41}$$

According to the discussion at the end of section 4.3. the following two systems of field equations are equivalent

$$\left.\begin{aligned} \varepsilon_\mu &= - t_\mu \\ \sigma_{\alpha\beta} &= - s_{\alpha\beta} \end{aligned}\right\} \qquad (L_A = 0) \tag{5.42}$$

$$\left.\begin{aligned} G_\mu &= - T_\mu \\ \sigma_{\alpha\beta} &= - s_{\alpha\beta} \end{aligned}\right\} \qquad (L_A = 0) \tag{5.43}$$

But (5.43) can be written as

$$G'_\mu = - T'_\mu \qquad (\sigma^{\alpha\beta} = - s^{\alpha\beta} \ , \ L_A = 0) \tag{5.44}$$

Furthermore, using the results of the sections 5.2. and 5.3., we have

$$G'_{\mu\nu} = G'_{\nu\mu} \tag{5.45}$$

$$T'_{\mu\nu} = T'_{\nu\mu} \qquad (L_A = 0 \ , \ \sigma^{\alpha\beta} = - s^{\alpha\beta}) \tag{5.46}$$

$$\overset{LC}{D} G'_\mu = 0 \tag{5.47}$$

$$\overset{LC}{D} T'_\mu = 0 \qquad (L_A = 0 \ , \ \sigma^{\alpha\beta} = - s^{\alpha\beta}) \tag{5.48}$$

We call the system (5.44) --------- (5.48) the Levi-Cività equations associated with L_{tot}. They are unique and reduce, in the absence of torsion, to the Levi-Cività equations of sections 5.2. and 5.3. In the the absence of torsion the matter equations $\sigma^{\alpha\beta} = - s^{\alpha\beta}$ disappear.

Remarks

The Levi-Cività equations associated with L_{tot} illustrate that it is indeed reasonable to treat the contorsion (and thus the torsion $\theta^{\alpha} = \tau^{\alpha}{}_{\beta} \wedge \theta^{\beta}$) as a matter field.

In the example (5.33) the equations

$$\sigma^{\alpha\beta} = - s^{\alpha\beta} \tag{5.49}$$

read as

$$D\star\Omega^{\alpha\beta} = s^{\alpha\beta} \tag{5.50}$$

Since (5.50) contains the pure gravitational term $\overset{LC}{D}\star\overset{LC}{\Omega}{}^{\alpha\beta}$ one could argue that (5.50) should be considered as an equation for gravity rather than as an equation for the matter fields. This objection is, however, not justified since there will always be gravitational terms in a matter field equation, provided L_{tot} contains mixed terms of the respective matterfields with the (Levi-Cività) curvature.

We conclude that it is a question of taste to consider gravitation theories with dynamical torsion as SL(2,$\mathbb{C}$)-gauge theories or not. But as long as there is no physical evidence for the existence of torsion in nature, the controversy about gravity admitting a gauge field description or not is of an esoteric character.

VI Examples of Gravitation Theories *

The appearance of singularities and the non-renormalizability of general relativity have motivated several authors to develop alternative theories of gravitation. The non-abelian YM gauge theories are, by contrast, renormalizable. For this reason people originally expected that a gauge formulation of gravitation should lead to a renormalizable theory of gravity. Some of the proposals are discussed in this and in the following chapter. Most of the alternative theories are, however, afflicted with serious difficulties on the classical level, but apart from this there exists, so far , still no renormalizable (gauge) theory of gravitation. The examples below are analysed within the general framework developed in the chapters IV and V .

In section 6.1. and 6.2. we start with the original gauge version of Utiyama and Kibble. The YM-type Lagrangian for gravitation is analysed in section 6.3. A modified YM gauge theory of gravitation , proposed by Fairchild Jr. , is discussed in section 6.4. In section 6.5. we present the translational gauge concept of Y.M. Cho . The probably most doubtful proposal, namely the gauge version of M. Carmeli , is briefly discussed in section 6.6.

6.1. The Utiyama approach

In the first part of the famous paper by Utiyama {47} the concept of a non-abelian gauge theory is generalized to arbitrary compact groups. In section 4 of that article the author tries to incorporate gravity. He starts with a dynamical Lorentz frame and a dynamical metric connection, but torsion is abandoned later and the resulting analysis covers exclusively the pure Levi-Cività theories with a dynamical Lorentz frame, as we have worked out in the sections 5.2., 5.3., and 5.4.

* See also the Refs. {13,14,55,56}

6.2. The Einstein-Cartan theory

A consequent discussion incorporating dynamical torsion was given later by Kibble {26}. The framework of Kibble is covered by our analysis in chapter IV. For instance, in Ref.{26} one finds an alternative deduction of the differential identities (4.55), (4.56), (4.57), and (4.58). The local Lorentz invariance and the diffeomorphism invariance are presented exclusively in terms of infinitesimals.

By contrast, a modern presentation of the Einstein-Cartan theory and of the structures involved was given by A.Trautman {46}. Actually chapter IV is heavily footing on the analysis in Ref.{46}. An extensive list of further references is given in the review on " General relativity with spin and torsion" by Hehl et al. (see Ref.{16}).

The field Lagrangian is (see remark (in brackets) in sc.4.3.)

$$L_g = 1/2\ \eta_{\alpha\beta} \wedge \Omega^{\alpha\beta} \tag{6.1}$$

The associated currents (4.12) and (4.13) are

$$\varepsilon_\mu = \eta_{\mu\alpha\beta} \wedge \Omega^{\alpha\beta} = \eta_{\mu\alpha\beta} \wedge \overset{LC}{\Omega}{}^{\alpha\beta} + \eta_{\mu\alpha\beta} \wedge (\overset{LC}{D}\tau^{\alpha\beta} + \tau^{\alpha}{}_{\lambda} \wedge \tau^{\lambda\beta}) \tag{6.2}$$

$$\sigma^{\mu\nu} = D\eta^{\mu\nu} = (\tau^{\mu}{}_{\lambda} \wedge \eta^{\lambda\nu} + \tau^{\nu}{}_{\lambda} \wedge \eta^{\mu\lambda}) \tag{6.3}$$

The Cartan equations

$$\sigma^{\mu\nu} = - s^{\mu\nu} \tag{6.4}$$

are purely algebraic and have the unique solution for the torsion

$$Q^{\mu}_{\alpha\beta} = - (J^{\mu}{}_{\alpha\beta} - 1/2\ \delta^{\mu}_{\alpha} J^{\lambda}{}_{\lambda\beta} - 1/2\ \delta^{\mu}_{\beta} J^{\lambda}{}_{\alpha\lambda}) \tag{6.5}$$

Consequently there is no torsion in empty space and matter produces torsion only in the sense of a contact interaction. Since the spin-density associated with scalar fields or with additional (nonlinear) gauge fields is equal to zero, such sources do not generate torsion. This is, in particular,true for the following matter fields :

(i) isentropic perfect fluids (see appendix I)
(ii) the electromagnetic field
(iii) the heavy intermediate vector bosons in the weak interactions
(iv) the gluons in QCD

⋮

etc.

For such sources the system (4.59) and (4.60) reduces to the ordinary Einstein equations

$$\eta_{\mu\alpha\beta}\wedge{}^{LC}\Omega^{\alpha\beta} = -t_\mu \tag{6.6}$$

In the case of $s_{\alpha\beta} \neq 0$ one can rewrite (4.59) as an Einstein equation with a modified right hand side

$$\eta_{\mu\alpha\beta}\wedge{}^{LC}\Omega^{\alpha\beta} = -t'_\mu \tag{6.7}$$

$$t'_\mu = t_\mu + \eta_{\mu\alpha\beta}\wedge({}^{LC}D\tau^{\alpha\beta} + \tau^\alpha{}_\lambda\wedge\tau^{\lambda\beta}) \tag{6.8}$$

From (6.5) we see that all the torsional parts in (6.8) may be expressed in terms of the spin-density of the (non-torsional) matter fields.

As a consequence of (6.7) one has

$$t'_{\mu\nu} = t'_{\nu\mu} \quad \text{and} \quad {}^{LC}D t'_\mu = 0 \tag{6.9}$$

But (6.9) is already a consequence of (6.4) and of $L_A = 0$. In order to give a proof for this , let us derive the Levi-Cività (field)-equations: The decomposition (5.35) is realized by

$$L'_g = 1/2\ \eta_{\alpha\beta}\wedge{}^{LC}\Omega^{\alpha\beta} \tag{6.10}$$

$$L'_m = 1/2\ \eta_{\alpha\beta}\wedge({}^{LC}D\tau^{\alpha\beta} + \tau^\alpha{}_\lambda\wedge\tau^{\lambda\beta}) + L_m \tag{6.11}$$

Since $\Delta_\theta L'_g = \delta\theta^\mu\wedge\eta_{\mu\alpha\beta}\wedge{}^{LC}\Omega^{\alpha\beta} + 1/2\ \eta_{\alpha\beta}\wedge{}^{LC}D\delta\omega^{LC}\alpha\beta$

$= \delta\theta^\mu\wedge\eta_{\mu\alpha\beta}\wedge{}^{LC}\Omega^{\alpha\beta}$, one has

$$E'_\mu = \eta_{\mu\alpha\beta}\wedge{}^{LC}\Omega^{\alpha\beta} \tag{6.12}$$

Similarly one finds with (6.4) and with $L_A = 0$ that

$$T'_\mu = t'_\mu \tag{6.13}$$

Notice that (6.13) relies on the fact that L'_m <u>does not</u> contain any mixed terms of ${}^{LC}\Omega^\alpha{}_\beta$ with the contorsion as in (5.38).

The non-dynamical character of torsion in the Einstein-Cartan theory has disturbed a lot of people and caused some of them to develop alternative theories (see also chapter VII). In the previous chapter, however, we have worked out the point of view that torsion can always be regarded and treated as a matter field.

In the sense of a Poincaré gauge field theory it is correct to say that $\Theta^\mu = D\theta^\mu$ is as much a translational field strength as $\Omega^\alpha{}_\beta = D\omega^\alpha{}_\beta$ is a rotational field strength. But, because $\Theta^\mu = \tau^\mu{}_\nu \wedge \theta^\nu$, the relation between $Q^\mu_{\alpha\beta}$ and $\tau^\mu_{\alpha\beta}$ is purely algebraic. So what is then the physically meaningful potential for torsion? Inspired by the ideas of Hojman et al.{21} , we may introduce a potential $\Phi^\alpha{}_\beta$ for $\tau^\alpha{}_\beta$, i.e. let $\Phi^\alpha{}_\beta$ be a tensor 1-form of type so(1,3) and

$$\tau^\alpha{}_\beta = \overset{LC}{D}\Phi^\alpha{}_\beta \tag{6.14}$$

The Lagrangian (6.1) then reads as

$$L_g = 1/2\ \eta_{\alpha\beta} \wedge \overset{LC}{\Omega}{}^{\alpha\beta} + 1/2\ \eta_{\alpha\beta} \wedge \overset{LC}{D}\Phi^\alpha{}_\lambda \wedge \overset{LC}{D}\Phi^\lambda{}_\beta + 1/2\ \eta_{\alpha\beta} \wedge \overset{LC}{D}{}^2\Phi^{\alpha\beta} \tag{6.15}$$

The field equations (6.7) remain unchanged, but the Cartan equations become dynamical, i.e.

$$1/2\ \overset{LC}{D}{}^2\Phi^\mu{}_\lambda \wedge \eta^{\lambda\nu} + 1/2\ \overset{LC}{D}{}^2\Phi^\nu{}_\lambda \wedge \eta^{\mu\lambda} = - \overset{LC}{D}s^{\mu\nu} \tag{6.16}$$

Equation (6.16) shows a strange behaviour: The left hand side is linear in the components of the Levi-Cività curvature.Hence it vanishes identically on Minkowski space and any function $\Phi^\alpha{}_\beta$ is a vacuum solution of (6.16) on a flat space-time. But in the presence of sources, with $J^{\lambda\mu\nu}{}_{|\lambda} \neq 0$, the $\Phi^\alpha{}_\beta$-field forces space-time to be curved.

Remarks: One may drop the exact differential $1/2\ d(\eta_{\alpha\beta} \wedge \tau^{\alpha\beta})$ in (6.1) and consider the total Lagrangian

$$\hat{L}_{tot} = 1/2\ \eta_{\alpha\beta} \wedge (\overset{LC}{\Omega}{}^{\alpha\beta} + \tau^\alpha{}_\omega \wedge \tau^{\omega\beta}) + L_m \tag{6.17}$$

It is clear how $\hat{L}_{tot}$ defines a pure Levi-Cività theory. Conversely $\hat{L}_{tot}$ has no natural place in the gauge theoretic framework of cpt. IV. This illustrates the fact that gauge theories of gravitation can be considered as a subclass of pure Levi-Cività theories.The system (5.31) and (6.3;6.4) associated with the above $\hat{L}_{tot}$ is different from the system (4.59) and (6.3 ; 6.4) associated with $\hat{L}_{tot}$.

6.3. The YM-type Lagrangian for gravitation [11 29 34 55]

The currents (4.12) and (4.13) associated with

$$L_g = -1/2\ \Omega^{\alpha}_{\ \beta} \wedge {}^*\Omega_{\alpha}^{\ \beta} \tag{6.18}$$

are

$$\varepsilon_{\mu} = 1/2\ i_{\xi_\mu} \Omega^{\alpha}_{\ \beta} \wedge {}^*\Omega_{\alpha}^{\ \beta} - 1/2\ \Omega^{\alpha}_{\ \beta} \wedge i_{\xi_\mu} {}^*\Omega_{\alpha}^{\ \beta} \tag{6.19}$$

$$\sigma^{\alpha\beta} = D{}^*\Omega^{\alpha\beta} \tag{6.20}$$

As we have noticed in section 4.2. this Lagrangian satisfies the strong local Lorentz invariance. Any substitution of the form

$$\tau^{\alpha}_{\ \beta} \longrightarrow \tilde{\tau}^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \rho}\{ {}^{LC}\omega^{\rho}_{\ \sigma} + \tau^{\rho}_{\ \sigma}\}\ \bar{\Lambda}^{-1}{}^{\sigma}_{\ \beta} - d\Lambda^{\alpha}_{\ \rho} . \bar{\Lambda}^{-1}{}^{\rho}_{\ \beta} - {}^{LC}\omega^{\alpha}_{\ \beta} \tag{6.21}$$

with an arbitrary $L^{\uparrow}_{+}$-valued function $\Lambda^{\alpha}_{\ \beta}(x)$ causes simply a Lorentz transformation

$$\Omega^{\alpha}_{\ \beta} \longrightarrow \Lambda^{\alpha}_{\ \lambda} \cdot \bar{\Lambda}^{-1}{}^{\rho}_{\ \beta} \cdot \Omega^{\lambda}_{\ \rho} \tag{6.22}$$

of the internal indices α and β of the curvature . Notice that the Lorentz frames and the associated Levi-Civitä connection are kept fixed. The substitution (6.21) simply absorbes a gauge transformation of $\omega^{\alpha}_{\ \beta}$ in a redefinition of the local components of $\tau^{\alpha}_{\ \beta}$.

In the case of non-spinning matter, the field equations are

$$\varepsilon_{\mu} = -\ t_{\mu} \tag{6.23}$$

$$\sigma_{\alpha\beta} = 0 \tag{6.24}$$

Since the curvature indices α and β appearing in equation (6.19) are contracted, a substitution (6.21) does not affect the equation (6.23) ; there is no torsion on the right hand side of this equation. Similarly a substitution (6.21) does not change equation (6.24) either.

Apart from this disease another difficulty follows from the fact that ε_{μ} is not only identically symmetric ($D\sigma^{\alpha\beta} = 0$)

but also traceless. Hence, if torsion is neglected ($\tau^{\alpha\beta} = 0$), the resulting equations

$$L_{\xi}C_{\mu} = -t_{\mu} \tag{6.25}$$

$$L_{\sigma}C_{\alpha\beta} = -s_{\alpha\beta} \tag{6.26}$$

are still unphysical: they do not admit a non-relativistic isentropic perfect fluid ($\rho > 3p$) as a source for gravity.

Taken all together shows that (6.18) is not a candidate for a gravitational Lagrangian. A complete different situation arises, however, if (6.18) is just one component of L_g, i.e. if the Lagrangian is , for instance, of the type

$$L_g = 1/2\ \eta_{\alpha\beta} \wedge \Omega^{\alpha\beta} - \ell^2/K\ 1/2\ \Omega^{\alpha}{}_{\beta} \wedge {}^*\Omega_{\alpha}{}^{\beta} \tag{6.27}$$

The Lagrangian (6.27) does not satisfy the strong local Lorentz invariance.(ℓ is the Planck length and K is a dimensionless coupling constant.)

6.4. The gauge theory of Fairchild, Jr [10] [55]

Apart from global questions any metric connection $\omega = 1/2\ \omega_{\alpha\beta}\ B^{\alpha\beta}$ may be lifted to a $sl(2,\mathbb{C})$-valued spinor connection $\mathfrak{w} = 1/2\ \omega_{\alpha\beta}\ \mathfrak{B}^{\alpha\beta}$ (see section 2.3.).

Let Ψ be a Dirac-spinor , i.e. Ψ is a tensor o-form of type (S, E_{Dirac})

$$S = D^{(1/2,0)} \oplus D^{(0,1/2)} \tag{6.28}$$

E_{Dirac} : Dirac spin space

Since $S_*(\mathfrak{B}^{\alpha\beta}) = 1/2\ \gamma^{\alpha\beta}$ ($\gamma^{\alpha\beta} = 1/2\ [\gamma^{\alpha}, \gamma^{\beta}]$; γ^{μ}:Dirac matrices), we conclude

$$D\Psi = d\Psi + 1/4\ \omega_{\alpha\beta}\ \gamma^{\alpha\beta}\ \Psi$$

Similarly the curvature $\Omega = 1/2\ \Omega^{\alpha}{}_{\beta}\ B_{\alpha}{}^{\beta}$ associated with ω may be considered as $sl(2,\mathbb{C})$-valued, namely

$$\tilde{\Omega} = 1/2\ \Omega^{\alpha}{}_{\beta}\ \tilde{B}_{\alpha}{}^{\beta}$$

We define the "Dirac spin space curvature"

$$1/2\ \Phi_{\mu\nu}\ \theta^{\mu}\wedge\theta^{\nu} = \Phi = S_{*}(\tilde{\Omega}) \tag{6.29}$$

The Lagrangian given by Fairchild,Jr.(Ref.{10}) is

$$L_g = -\mathcal{L}_g\ \eta$$

$$\mathcal{L}_g = -1/4\ \bar{\lambda}^2\ \mathrm{Tr}(\Phi_{\mu\nu}\Phi^{\mu\nu}) + 1/2\ \mu^2\ \bar{\lambda}^2\ \mathrm{Tr}(\Phi_{\mu\nu}\gamma^{\mu\nu}) \tag{6.30}$$

Since $\Phi_{\mu\nu} = 1/4(R_{\alpha\beta})_{\mu\nu}\ \gamma^{\alpha\beta}$ and $\mathrm{Tr}(\gamma^{\alpha\beta}\gamma^{\rho\sigma}) = 4\ g^{\rho\beta}g^{\sigma\alpha} - 4\ g^{\rho\alpha}g^{\sigma\beta}$ (6.31)

we find

$$\mathrm{Tr}(\Phi_{\mu\nu}\Phi^{\mu\nu}) = -\ 8\ R_{\alpha\beta\mu\nu}\ R^{\alpha\beta\mu\nu} \tag{6.32}$$

Similarly we have

$$\mathrm{Tr}(\Phi_{\mu\nu}\gamma^{\mu\nu}) = R_{\alpha\beta\mu\nu}(g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}) = -\ 2\ R \tag{6.33}$$

Hence

$$\mathcal{L}_g = 2\ \bar{\lambda}^2\ R_{\alpha\beta\mu\nu}\ R^{\alpha\beta\mu\nu} - \mu^2\ \bar{\lambda}^2\ R \tag{6.34}$$

respectively

$$L_g = -\ \bar{\lambda}^2\ \Omega^{\alpha}{}_{\beta}\wedge{}^{*}\Omega_{\alpha}{}^{\beta} + \mu^2\ \bar{\lambda}^2\ 1/2\ \eta_{\alpha\beta}\wedge\Omega^{\alpha\beta} \tag{6.35}$$

This is our old friend (6.27). Since Fairchild places great importance to the distinction between (linear metric) connections on the Dirac spin space and (linear metric) connections on the tangent space , he gives (6.35) in the unfamiliar (and also unpractical) form (6.30). Such a distinction is, however, completely unnecessary, as we have seen in chapter II. Neither from the point of view of mathematics nor from the point of view of physics there is any reason why one should write the Lagrangian (6.35) in terms of the Dirac spin space curvature and in terms of Dirac γ-matrices.

We rewrite (6.35) in the form (6.27)

$$L_g = - \ell^2/K \; 1/2 \; \Omega^{\alpha}{}_{\beta} \wedge {}^*\Omega_{\alpha}{}^{\beta} + 1/2 \; \eta_{\alpha\beta} \wedge \Omega^{\alpha\beta} \qquad (6.36)$$

($K = 1/2 \; \ell^2 \mu^2$, ℓ: Planck length)

We neglect torsion and consider the Levi-Cività equation (5.31) associated with (6.36):

$$G_{\mu} = - T_{\mu} \qquad (6.37)$$

$$G_{\mu} = \eta_{\mu\alpha\beta} \wedge {}^{LC}\Omega^{\alpha\beta} + \ell^2/K \; 1/2 \; (i_{\mu} {}^{LC}\Omega^{\alpha}{}_{\beta} \wedge {}^*{}^{LC}\Omega_{\alpha}{}^{\beta} - {}^{LC}\Omega^{\alpha}{}_{\beta} \wedge i_{\mu} {}^*{}^{LC}\Omega_{\alpha}{}^{\beta}) +$$

$$+ \ell^2/K \; \{\eta_{\nu\lambda} \wedge {}^{LC}D {}^*(\theta^{\lambda} \wedge {}^{LC}D {}^*{}^{LC}\Omega_{\mu}{}^{\nu}) - 1/2 \; \eta_{\lambda\nu} \wedge {}^{LC}D {}^*(\theta_{\mu} \wedge D {}^*{}^{LC}\Omega^{\lambda\nu})\} \qquad (6.38)$$

It is easy to see that $G_{\mu} = 0$ contains the Schwarzschild solution. However, let $h_{\alpha\beta}$ be a perturbation of the flat metric $\hat{\eta}_{\alpha\beta}$, i.e. $g_{\alpha\beta} = \hat{\eta}_{\alpha\beta} + h_{\alpha\beta}$. Apart from terms $\Box h_{\alpha\beta}$ the linearisation of G_{μ} then also yields terms $\Box\Box h_{\alpha\beta}$, which cannot be gauged away. (See for this point also Ref.{44}).Consequently the equation (6.37) is not physical, unless K is very large compared to ℓ.

In his discussion Fairchild ignores the last term in (6.38). The linearised approximation then comes out well. But there arises a new problem with the differential identities : The right hand side is subject to ${}^{LC}D T_{\mu} = 0$ but this will, in general, not be a consequence of the field equations any more.

Our final conclusion is that (6.27) is not a candidate for a gravitational Lagrangian.

6.5. The translational gauge concept of Y.M. Cho

Let ξ_i denote a Lorentz frame of vectorfields and θ^j the associated dual frame of 1-forms. We may expand ξ_i in a coordinate frame $\partial_{\mu} = \partial/\partial x^{\mu}$

$$\xi_i = h_i^\mu \, \partial_\mu \tag{6.39}$$

The commutator functions T_{ij}^k defined by

$$[\xi_i, \xi_j] = T_{ij}^k \, \xi_k \tag{6.40}$$

may be expressed in terms of the h_i^μ and their derivatives

$$T_{ij}^k = (\partial_{\xi_i} h_j^\mu - \partial_{\xi_j} h_i^\mu) \, h_\mu^k \tag{6.41}$$

Now let $B = B^\mu \, e_\mu$ be a translational gauge potential ; e_μ is the canonical basis of $\mathbb{R}^4$. A local gauge transformation of B is given by

$$B \longrightarrow B' = B + d\Lambda \tag{6.42}$$

with an arbitrary $\mathbb{R}^4$-valued function Λ. The curvature 2-form is $G = dB$. Since $dB = dB_i \wedge \theta^i + B_i \, d\theta^i =$

$= 1/2 \, (\partial_j B_i - \partial_i B_j) \, \theta^j \wedge \theta^i - 1/2 \, T_{rs}^i \, \theta^r \wedge \theta^s \, B_i$, we conclude

$$G_{ij} = \partial_i B_j - \partial_j B_i - T_{ij}^k \, B_k \tag{6.43}$$

$(\partial_i B_j = \partial_{\xi_i} B_j)$

Notice that

$$G'_{ij} = G_{ij} \tag{6.44}$$

under local gauge transformations (6.42).

The crucial point of the approach in Ref.{ 7} is the so-called Kibble identification

$$h_i^\mu = \delta_i^\mu + B_i^\mu \tag{6.45}$$

which connects the gauge potential B with the Lorentz frames ξ_i respectively θ^j . Any variation δB of the gauge potential induces a variation $\delta\xi_i$ (respectively $\delta\theta^i$) of the Lorentz frames and vice versa. One runs into troubles, however, if one applies local gauge transformations (6.42) to (6.45) : In order to keep the Lorentz metric (defined by ξ_i) fixed, Cho absorbes local gauge transformations of B in local coordinate transformations (respectively local diffeomorphisms).We do not want to repeat that cumbersome point here.

By (6.41) and (6.45) the coefficients of the curvature G may be identified with the commutator functions, i.e.

$$G^{\alpha}_{ij} = T^{\alpha}_{ij} \tag{6.46}$$

and hence

$$d\theta^{k} = -1/2\, G^{k}_{ij}\, \theta^{i}\wedge\theta^{j} \tag{6.47}$$

As Cho proves, the only locally Lorentz invariant Lagrangian quadratic in the coefficients G^{k}_{ij} is given by

$$L_E = -1/2\,\{1/4\, G^{i}_{ab}\, G^{\cdot ab}_{i\cdot\cdot} + 1/2\, G^{\cdot asr}_{\cdot sar} G^{\cdot\cdot}_{\cdot\cdot} - G^{i}_{ia}\, G^{j\ a}_{j\ \cdot}\}\,\eta =$$

$$= -1/2\,(d\theta^{i}\wedge\theta^{j})\wedge *(d\theta_{j}\wedge\theta_{i}) + 1/4\,(d\theta^{i}\wedge\theta_{i})\wedge *(d\theta^{j}\wedge\theta_{j}) \tag{6.48}$$

Actually, this is the Hilbert-Einstein Lagrangian (5.33). A straightforward proof is given in appendix III.

The total Lagrangian is

$$L_{tot} = L_E + L_m \tag{6.49}$$

The a priori dynamical element is the gauge potential B. But since variations δB induce variations $\delta\theta^{\mu}\{\delta B\}$ and vice versa, it is more practical to consider θ^{i} as the dynamical element. At this point one may object that L_E contains exclusively the frames θ^{i} and their derivatives $d\theta^{i}$. By contrast, a general matter Lagrangian L_m contains also covariant exterior derivatives

$$D\Psi = d\Psi + S_{*}(\overset{LC}{\omega})\wedge\Psi \tag{6.50}$$

The solution of the Cartan structure equations

$$\overset{LC}{\omega}_{ij} + \overset{LC}{\omega}_{ij} = 0 \tag{6.51}$$

$$d\theta^{i} + \overset{LC}{\omega}{}^{i}{}_{j}\wedge\theta^{j} = 0 \tag{6.52}$$

is, however, given by

$$\overset{LC}{\omega}_{ij} = -1/2\,\theta^{k} *\{d\theta_{i}\wedge\eta_{jk} - d\theta_{j}\wedge\eta_{ik} - d\theta_{k}\wedge\eta_{ij}\} \tag{6.53}$$

(See also section 2.5.) Consequently L_m may always be expressed exclusively in terms of $\Psi, d\Psi, \theta^{i}$ and $d\theta^{i}$. The equations resulting from variations of (6.49) with respect to θ^{i} are the Einstein field equations.

As we have mentioned in section 5.4. , general relativity may formally be considered as a gauge theory of the translation group T^4 .

The alternative approach of Cho is in so far intriguing, as L_E appears as a quadratic Lagrangian in the coefficients of the curvature G. The drawback, however , is the fact that the Kibble identification (6.45) is not properly compatible with the local T^4- gauge invariance : Local diffeomorphisms must be taken into account in order to save the idea. But such a procedure has nothing in common with the concept of an internal symmetry any more.

6.6. The gauge concept of M. Carmeli [2] [3]

With the help of the Bianchi identities we recast the Einstein equations into a new form. The Ricci tensor 1-forms R_α associated with a Levi-Civitä curvature ${}^{LC}\Omega^\alpha{}_\beta = 1/2\ R^\alpha{}_{\beta\mu\nu}\theta^\mu\wedge\theta^\nu$ are defined by

$$R_\alpha = R^\lambda_{\alpha\lambda\nu}\ \theta^\nu = R_{\alpha\nu}\ \theta^\nu \tag{6.54}$$

The curvature scalar is

$$R = *(\eta^\alpha\wedge R_\alpha) \tag{6.55}$$

The Weyl 2-forms are

$$C^{\alpha\beta} = {}^{LC}\Omega^{\alpha\beta} + 1/6\ R\ \theta^\alpha\wedge\theta^\beta - 1/2\ (\theta^\alpha\wedge R^\beta - \theta^\beta\wedge R^\alpha) \tag{6.56}$$

It is well known that the Bianchi identities

$${}^{LC}D\ {}^{LC}\Omega^\alpha{}_\beta = 0 \tag{6.57}$$

can equivalently be rewritten as a YM-type equation

$${}^{LC}D{*C}^{\alpha\beta} = \{R^{\rho[\alpha|\beta]} + 1/6\ g^{\rho[\beta}R^{|\alpha]}\}\eta_\rho \tag{6.58}$$

With the Einstein field equations

$$R_{\mu\nu} = T_{\mu\nu} - 1/2\ g_{\mu\nu}\ T \tag{6.59}$$

we can express the right hand side of (6.58) in terms of the matter (see also Ref.{15}), i.e.

$$ {}^{LC}D *C^{\alpha\beta} = \mathcal{J}^{\alpha\beta} \tag{6.60} $$

where $\mathcal{J}^{\alpha\beta} = \mathcal{J}^{\lambda\alpha\beta}\,\eta_\lambda$

and $$ \mathcal{J}^{\lambda\alpha\beta} = T^{\lambda[\alpha|\beta]} - 1/3\, T^{[\beta}\, g^{\alpha]\lambda} \tag{6.61} $$

The Bianchi identities (6.57) follow from a variational principle in the following way:

The independent dynamical fields are taken to be a $so(1,3)$-valued tensor 2-form ${}^{LC}\Omega = 1/2\,{}^{LC}\Omega^{\alpha}{}_{\beta}\, B_{\alpha}{}^{\beta}$ and, in addition, the Lorentz frames θ^ν. The Levi-Cività connection can be given exclusively in terms of θ^μ and $d\theta^\mu$:

$$ {}^{LC}\omega_{\mu\nu} = \frac{-1}{2}\,\theta^\lambda * \{d\theta_\mu \wedge \eta_{\nu\lambda} - d\theta_\nu \wedge \eta_{\mu\lambda} - d\theta_\lambda \wedge \eta_{\mu\nu}\} \tag{6.62} $$

Any variation $\delta\theta^\mu$ induces a variation $\delta\,{}^{LC}\omega^{\mu}{}_{\nu}$ given by (2.35). The Lagrangian is

$$ L = {}^{LC}\Omega_{\alpha}{}^{\beta} \wedge \{1/2\,{}^{LC}\Omega^{\alpha}{}_{\beta} - d\,{}^{LC}\omega^{\alpha}{}_{\beta} - {}^{LC}\omega^{\alpha}{}_{\lambda} \wedge {}^{LC}\omega^{\lambda}{}_{\beta}\} \tag{6.63} $$

The variations $\delta\,{}^{LC}\Omega^{\alpha}{}_{\beta}$ yield the equation

$$ {}^{LC}\Omega^{\alpha}{}_{\beta} = d\,{}^{LC}\omega^{\alpha}{}_{\beta} + {}^{LC}\omega^{\alpha}{}_{\lambda} \wedge {}^{LC}\omega^{\lambda}{}_{\beta}. \tag{6.64} $$

which says that ${}^{LC}\Omega^{\alpha}{}_{\beta}$ is the Levi-Cività curvature associated with the Lorentz frames θ^μ. The variations $\delta\theta^\mu$ finally lead to the Bianchi identity (6.57).

Although (6.60) looks very much like a YM field equation , such a procedure does not seriously deserve to be considered as a true Lagrangian field theory of gravitation. In his discussion Carmeli even fails to put the Levi-Cività conditions on ${}^{LC}\omega^{\alpha}{}_{\beta}$ in a dynamical form, as we did by treating θ^μ (and not ${}^{LC}\omega^{\alpha}{}_{\beta}$) as the second dynamical element. Writing the above formalism consequently in terms of spinors neither clears up the weak points nor helps to disguise them.

VII The Poincaré Gauge Theory of Hehl et al.[17] [18] [19] [20]

The gravitational field equations of the theories discussed in the sections 6.1.,6.2.,6.5., and 6.6. reduce, in the case of Newtonian matter, identically to the Einstein equations. Consequently, these theories are not essentially different from general relativity.

By contrast, the "Poincaré gauge field theory" of Hehl et al.(see Refs{17,18}) deserves to be considered as a true alternative gravitation theory for several reasons. The field Lagrangian is the sum of a translational Lagrangian (quadratic in torsion) and of the rotational Lagrangian (6.18). The YM-type part appears only in the presence of microscopic spinning matter. For a vanishing spin-density of the matter, Hehl et al. work in the so-called teleparallel limit. This means that the dynamics of gravity can be described by particular Lorentz frames θ^{ν} for which the relation $\theta^{\mu} = d\theta^{\mu}$ holds. The torsion is no longer an independent dynamical element.

Although this theory relies on a completely different philosophy, it is observationally indistinguishable from general relativity.

In section 7.1. we present the field Lagrangian and compare the teleparallel limit of the translational part to the Hilbert-Einstein Lagrangian.

In section 7.2. we prove that the first post-Newtonian approximation of the field equations (in the teleparallel limit) is identical to that of general relativity.

Conservation laws for asymptotically flat geometries are derived in section 7.3.

In section 7.4. we adapt the method of Epstein and Wagoner (see Ref.{ 9 }) and show that the post-Newtonian generation of gravitational radiation is the same as in general relativity.

We close this chapter with some final conclusions in section 7.5.

7.1. The teleparallel limit

The field Lagrangian is the sum of a translational and a rotational part:

$$L_g = L_{transl} + L_{rot} \tag{7.1}$$

$$L_{transl} = -1/2\ \{(\theta^\alpha \wedge \theta^\beta) \wedge^* (\theta_\beta \wedge \theta_\alpha) - \lambda/2\ (\theta^\alpha \wedge \theta_\alpha) \wedge^* (\theta^\beta \wedge \theta_\beta)\} \tag{7.2}$$

$$L_{rot} = -\ell^2/2k\ \Omega^{\alpha\beta} \wedge^* \Omega_{\alpha\beta} \tag{7.3}$$

The currents (4.12) and (4.14) associated with (7.1) are

$$\varepsilon^\mu = \varepsilon^\mu_E + (\lambda - 1)\ \Delta\varepsilon^\mu + \varepsilon^\mu_R \tag{7.4}$$

$$\varepsilon^\mu_E = -\ D\{\theta^\nu \wedge^* (\theta_\nu \wedge \theta^\mu)\} - \theta^\nu \wedge^* (\theta^\mu \wedge \theta_\nu) + \\ + 1/2\ D\{\theta^\mu \wedge^* (\theta^\nu \wedge \theta_\nu)\} + 1/2\ \theta^\mu \wedge^* (\theta^\nu \wedge \theta_\nu) \tag{7.5}$$

$$\Delta\varepsilon^\mu = \theta^\mu \wedge^* (\theta^\nu \wedge \theta_\nu) - 1/2\ \theta^\mu \wedge D^* (\theta^\nu \wedge \theta_\nu) \tag{7.6}$$

$$\varepsilon^\mu_R = \frac{\ell^2}{k}\ 1/2\ (i_\mu \Omega^\alpha{}_\beta \wedge^* \Omega_\alpha{}^\beta - \Omega^\alpha{}_\beta \wedge i^\mu * \Omega_\alpha{}^\beta) \tag{7.7}$$

$$\sigma^{\alpha\beta} = \frac{\ell^2}{k}\ D^*\Omega^{\alpha\beta} + \lambda\theta^\beta \wedge \theta^\alpha \wedge^* (\theta^\lambda \wedge \theta_\lambda) + 2\ \theta^\alpha \wedge \theta^\lambda \wedge^* (\theta_\lambda \wedge \theta^\beta) \tag{7.8}$$

Here ℓ and k are as in (6.27) and λ is a dimensionless parameter. Hehl et al. use $\lambda = 0$. In what follows, however, we do not discuss the full system (4.59) and (4.60). According to the philosophy of Hehl et al. , the YM-type part L_{rot} in (7.1) is associated with the intrinsic spin of the matter. In the case of Newtonian matter ($s^{\alpha\beta} = 0$) there is no source for the rotational field strength $\Omega^\alpha{}_\beta$. Consequently, Hehl et al. choose

$$\Omega^\alpha{}_\beta = 0 \tag{7.9}$$

and drop L_{rot} in (7.1). The remaining dynamical element is the Lorentz frame θ^ν. The resulting theory has an obvious geometric interpretation in a Weitzenböck space. Relative to a teleparallel

tetrad the connection forms $\omega^{\alpha}{}_{\beta}$ also vanish and the exterior covariant derivative reduces to the ordinary exterior derivative. The torsion θ^{α} is then equal to $d\theta^{\alpha}$. Thus, we use the following total Lagrangian relative to an arbitrary (orthonormal) frame

$$L_{tot} = -1/2\ (\theta^{\alpha}\wedge\theta^{\beta})\wedge^{*}(\theta_{\beta}\wedge\theta_{\alpha}) + \lambda/4\ (\theta^{\alpha}\wedge\theta_{\alpha})\wedge^{*}(\theta^{\beta}\wedge\theta_{\beta}) + L_{m} \tag{7.10}$$

which reduces to

$$L_{tot} = -1/2\ (d\theta^{\alpha}\wedge\theta^{\beta})\wedge^{*}(d\theta_{\beta}\wedge\theta_{\alpha}) + \lambda/4\ (d\theta^{\alpha}\wedge\theta_{\alpha})\wedge^{*}(d\theta^{\beta}\wedge\theta_{\beta}) + L_{m} \tag{7.11}$$

for a teleparallel frame.

Now we notice that for $\lambda = 1$ the gravitational Lagrangian in (7.11) is the Hilbert-Einstein Lagrangian (6.48). For $\lambda \neq 1$ this Lagrangian is not invariant under local Lorentz transformations

$$\theta^{\mu}(x) \longrightarrow \Lambda^{\mu}{}_{\nu}(x)\ \theta^{\nu}(x) \quad , \quad \Lambda(x) \varepsilon L_{+}^{\uparrow}$$

as was pointed out by Y.M. Cho (see section 6.5.). Variations of the teleparallel Lorentz frames lead to the field equations

$$\varepsilon_{E}^{\alpha} + (\lambda - 1)\ \Delta\varepsilon^{\alpha} = -\ t^{\alpha} \tag{7.12}$$

where now

$$\begin{aligned}\varepsilon_{E}^{\alpha} &= -\ d\{\theta^{\beta}\wedge^{*}(d\theta_{\beta}\wedge\theta^{\alpha})\} - d\theta^{\beta}\wedge^{*}(d\theta^{\alpha}\wedge\theta_{\beta}) + \\ &\quad + 1/2\ d\{\theta^{\alpha}\wedge^{*}(d\theta^{\beta}\wedge\theta_{\beta})\} + 1/2\ d\theta^{\alpha}\wedge^{*}(d\theta^{\beta}\wedge\theta_{\beta}) = \\ &= \eta^{\alpha}{}_{\mu\nu}\wedge\overset{LC}{\Omega}{}^{\mu\nu}\end{aligned} \tag{7.13}$$

$$\Delta\varepsilon^{\alpha} = d\theta^{\alpha}\wedge^{*}(d\theta^{\beta}\wedge\theta_{\beta}) - 1/2\ \theta^{\alpha}\wedge d^{*}(d\theta^{\beta}\wedge\theta_{\beta}) \tag{7.14}$$

ε_{E}^{α} in (7.13) is identically symmetric. By (4.55) the corresponding spin-density (see (7.8))

$$\sigma_{E}^{\alpha\beta} = \theta^{\beta}\wedge\theta^{\alpha}\wedge^{*}(d\theta^{\lambda}\wedge\theta_{\lambda}) + 2\ \theta^{\alpha}\wedge\theta^{\lambda}\wedge^{*}(d\theta_{\lambda}\wedge\theta^{\beta}) \tag{7.15}$$

is conserved , i.e.

$$d\sigma_{E}^{\alpha\beta} = 0 \tag{7.16}$$

The spin-density

$$(\lambda - 1)\ \theta^{\beta}\wedge\theta^{\alpha}\wedge *(d\theta^{\lambda}\wedge\theta_{\lambda}) \tag{7.17}$$

associated with $(\lambda - 1)\ \Delta\varepsilon^{\alpha}$ is , however, not identically conserved. But for Newtonian matter the source term in (7.12) is identically symmetric and thus, as a consequance of (7.12), we have

$$(\lambda - 1)\ D(\theta^{\beta}\wedge\theta^{\alpha}\wedge *(\theta^{\lambda}\wedge\theta_{\lambda})) \ = \ 0 \tag{7.18}$$

respectively in a teleparallel frame

$$(\lambda - 1)\ d(\theta^{\alpha}\wedge\theta^{\beta}\wedge *(d\theta^{\lambda}\wedge\theta_{\lambda})) \ = \ 0 \tag{7.19}$$

The symmetric part of $\Delta\varepsilon^{\alpha}$ is (in a teleparallel frame) given by

$$\Delta\varepsilon^{\alpha}_{S} \ = \ -1/2\ *\{(d\theta^{\alpha}\wedge\theta^{\beta} + d\theta^{\beta}\wedge\theta^{\alpha})\wedge *(d\theta^{\lambda}\wedge\theta_{\lambda})\}\ \eta_{\beta} \tag{7.20}$$

and therefore the equation (7.12) can equivalently be rewritten as the following system of two equations

$$\varepsilon^{\alpha}_{E} \ + \ (\lambda - 1)\ \Delta\varepsilon^{\alpha}_{S} \ = \ - t^{\alpha} \tag{7.21}$$

$$(\lambda - 1)\ d\{\theta^{\alpha}\wedge\theta^{\beta}\wedge *(d\theta^{\lambda}\wedge\theta_{\lambda})\} \ = \ 0 \tag{7.22}$$

The field equations (7.12) and the Einstein equations have a large family of common solutions. For any metric of the form

$$ds^2 \ = \ a_0^2(dx^0)^2 - a_1^2(dx^1)^2 - a_2^2(dx^2)^2 - a_3^2(dx^3)^2$$

with arbitrary functions a_i the forms $\Delta\varepsilon^{\alpha}$ vanish for the orthonormal basis

$$\theta^0 \ = \ a_0 dx^0\ ,\ \theta^1 \ = a_1 dx^1\ ,\ \theta^2 \ = a_2 dx^2\ ,\ \theta^3 \ = a_3 dx^3$$

because $d\theta^{\alpha}\wedge\theta_{\alpha} = 0$. This proves in particular that all spherically symmetric solutions of the Einstein equations are also solutions of (7.12) and thus a large body of astrophysical applications remains unchanged. The Kerr solution is, however , no vacuum solution of (7.12) for $\lambda \neq 1$. It is an open problem to generalize the stationary black hole solution to this case.

7.2. The first post-Newtonian approximation

In this section we prove that the linearized and the first post-Newtonian approximation of our system

$$\varepsilon_E^\alpha - (\lambda - 1)/2 \,*\{(d\theta^\alpha \wedge \theta^\beta + d\theta^\beta \wedge \theta^\alpha) \wedge *(d\theta^\gamma \wedge \theta_\gamma)\}\, \eta_\beta = - t^\alpha \tag{7.23}$$

$$(\lambda - 1)\, d\{\theta^\alpha \wedge \theta^\beta \wedge *(d\theta^\gamma \wedge \theta_\gamma)\} = 0 \tag{7.24}$$

is independent of λ, i.e. coincides with that of general relativity. For this purpose we expand the teleparallel frames θ^α in terms of a coordinate basis

$$\theta^\alpha = dx^\alpha + \Phi^\alpha{}_\beta \, dx^\beta \tag{7.25}$$

and decompose $\Phi_{\alpha\beta}$ into its symmetric and antisymmetric pieces

$$\Phi_{\alpha\beta} = \phi_{\alpha\beta} + a_{\alpha\beta} \tag{7.26}$$

$$\phi_{\alpha\beta} = 1/2\, \Phi_{(\alpha\beta)} \,, \qquad a_{\alpha\beta} = 1/2\, \Phi_{[\alpha\beta]} \tag{7.27}$$

Let us first consider the linearized approximations of (7.23) and (7.24). The second term on the left hand side of (7.23) contains obviously no linear terms. Furthermore, the antisymmetric field $a_{\alpha\beta}$ drops out identically in the linearized part of ε_E^α (as a consequence of the local Lorentz invariance of (6.48)). Thus equation (7.23) is reduced to the linearized Einstein equation for $\phi_{\alpha\beta}$. For $\lambda = 1$ (i.e. general relativity) equation (7.24) is empty; $a_{\alpha\beta}$ is just a gauge degree of freedom. For $\lambda \neq 1$ equation (7.24) leads in the linearized approximation to the decoupled source free equation

$$\Box\, a^{\alpha\beta} + a^{\lambda\alpha,\beta}{}_\lambda - a^{\lambda\beta,\alpha}{}_\lambda = 0 \tag{7.28}$$

for $a^{\alpha\beta}$, which is invariant under the gauge transformation

$$a_{\alpha\beta} \longrightarrow a_{\alpha\beta} + \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{7.29}$$

Imposing the gauge condition $a^{\alpha\beta}{}_{,\beta} = 0$, we are left with $\Box\, a^{\alpha\beta} = 0$. In the Newtonian approximation $a_{\alpha\beta}$ vanishes in this gauge and the theory gives the correct Newtonian limit.

Next we show that the first post-Newtonian approximation of (7.23) and (7.24) agrees with that of general relativity. (This was shown already in {37}, but the following discussion is simpler.) Since the term proportional to $(\lambda - 1)$ in (7.23) is of higher order in $\Phi_{\alpha\beta}$, it contains in the first post-Newtonian approximation at most quadratic expressions of the Newtonian approximation of $\phi_{\alpha\beta}$. But these vanish identically because $d\theta^{\alpha} \wedge \theta_{\alpha}$ is already quadratic in $\phi_{\alpha\beta}$. (The linear term vanishes identically.) Hence (7.23) reduces to the first post-Newtonian approximation of general relativity.

For the discussion of equation (7.24), we note first that the Newtonian approximation, $\phi^{(N)}_{\alpha\beta}$, of $\phi_{\alpha\beta}$ is diagonal in a suitable coordinate system: $\phi^{(N)}_{\alpha\beta} = \delta_{\alpha\beta}\, \phi$, ϕ = Newtonian potential. From this, one concludes easily that the quadratic terms in $\phi^{(N)}_{\alpha\beta}$ in (7.24) vanish identically and thus the first post-Newtonian approximation of (7.24) reduces to equation (7.28) for the post-Newtonian approximation, $a^{(PN)}_{\alpha\beta}$, of $a_{\alpha\beta}$. This implies that $a^{(PN)}_{\alpha\beta}$ vanishes also in a suitable gauge and hence our claim is proven.

Hehl and Nitsch {32} have shown that the post-post-Newtonian approximation does no more agree with that of general relativity. The deviations are, however , unmeasurably small.

7.3. Conservation laws

We start by writing the symmetric field equation (7.23) in arbitrary (not necessarily teleparallel or orthonormal) frames :

$$\varepsilon^{\alpha}_{E} - (\lambda - 1)/2 \,*\{(\theta^{\alpha}\wedge\theta^{\beta} + \theta^{\beta}\wedge\theta^{\alpha})\wedge *(\theta^{\gamma}\wedge\theta_{\gamma})\}\, \eta_{\beta} = - t^{\alpha} \qquad (7.30)$$

For ε^{α}_{E} we use the "Landau decomposition" derived in {43}:

$$\varepsilon^{\alpha}_{E} = \frac{1}{2\sqrt{-g}}\, d\{\sqrt{-g}\; {}^{LC}\omega_{\beta}{}^{\gamma} \wedge \eta^{\alpha\beta}{}_{\gamma}\} + t^{\alpha}_{L-L} \qquad (7.31)$$

Here t^{α}_{L-L} are the Landau-Lifschitz energy-momentum forms of the metric field given by

$$t^{\alpha}_{L-L} = -1/2\ \eta^{\alpha\beta\gamma\delta}(\overset{LC}{\omega}_{\sigma\beta}\wedge\overset{LC}{\omega}{}^{\sigma}{}_{\gamma}\wedge\theta_{\delta} - \overset{LC}{\omega}_{\beta\gamma}\wedge\overset{LC}{\omega}_{\sigma\delta}\wedge\theta^{\sigma}) \tag{7.32}$$

Relative to a coordinate basis $\theta^{\alpha} = dx^{\alpha}$ the 3-forms t^{α}_{L-L} are symmetric. Inserting (7.31) into (7.30) the symmetric field equation takes the form

$$-1/2\ d\{\sqrt{-g}\ \overset{LC}{\omega}_{\beta\gamma}\wedge\eta^{\alpha\beta\gamma}\} = \sqrt{-g}\ \tau^{\alpha} \tag{7.33}$$

where

$$\tau^{\alpha} = t^{\alpha} + t^{\alpha}_{L-L} - (\lambda - 1)\ \Delta t^{\alpha} \tag{7.34}$$

$$\Delta t^{\alpha} = 1/2\ *\{(\theta^{\beta}\wedge\theta^{\beta} + \theta^{\beta}\wedge\theta^{\alpha})\wedge*(\theta^{\gamma}\wedge\theta_{\gamma})\}\ \eta_{\beta} \tag{7.35}$$

It may be useful to note that the left hand side of (7.33) relative to a coordinate basis can be expressed in terms of the Landau-Lifschitz superpotential as follows :

$$d\{\sqrt{-g}\ \overset{LC}{\omega}_{\beta\gamma}\wedge\eta^{\mu\beta\gamma}\} = \sqrt{-g}\ H^{\mu\alpha\nu\beta}{}_{,\alpha\beta}\ \eta_{\nu}$$

where

$$H^{\mu\alpha\nu\beta} = \tilde{g}^{\mu\nu}\tilde{g}^{\alpha\beta} - \tilde{g}^{\mu\beta}\tilde{g}^{\nu\alpha}$$

with

$$\tilde{g}^{\mu\nu} = \sqrt{-g}\ g^{\mu\nu}$$

As in general relativity, the τ^{α} are interpreted as the total energy-momentum forms. By construction, they are symmetric relative to a coordinate basis:

$$\tau^{\alpha}\wedge dx^{\beta} = \tau^{\beta}\wedge dx^{\alpha} \tag{7.36}$$

From (7.33) we conclude that the field equations imply the conservation laws

$$d\{\sqrt{-g}\ \tau^{\alpha}\} = 0 \tag{7.37}$$

The last two equations imply

$$d\{\sqrt{-g}\ M^{\alpha\beta}\} = 0 \tag{7.38}$$

where

$$M^{\alpha\beta} = x^{\alpha}\tau^{\beta} - x^{\beta}\tau^{\alpha} \tag{7.39}$$

is the total angular momentum density.

For isolated systems the total momentum

$$P^{\alpha} = \int_{\Sigma} \sqrt{-g}\, \tau^{\alpha} \tag{7.40}$$

and the total angular momentum

$$J^{\alpha\beta} = \int_{\Sigma} \sqrt{-g}\, M^{\alpha\beta} \tag{7.41}$$

(Σ:spacelike surface) can be expressed with the help of the field equations (7.33) in terms of flux integrals at infinity:

$$P^{\alpha} = -1/2 \oint \sqrt{-g}\, {}^{LC}_{\omega}{}_{\beta\gamma} \wedge \eta^{\alpha\beta\gamma} \tag{7.42}$$

$$J^{\alpha\beta} = 1/2 \oint \sqrt{-g}\, \{(x^{\alpha}\eta^{\beta}{}_{\sigma\gamma} - x^{\beta}\eta^{\alpha}{}_{\sigma\gamma}) \wedge {}^{LC}_{\omega}{}^{\sigma\gamma} + \eta^{\alpha\beta}\} \tag{7.43}$$

As usual, the coordinates have to be asymptotically Lorentzian. P^{α} and $J^{\alpha\beta}$ transform like Lorentz tensors under coordinate transformations, which preserve this property. Clearly, the flux integrals (7.42) and (7.43) are the same as in general relativity. Since we have not seen equation (7.43) in the literature, we derive it in appendix IV .

7.4. Post-Newtonian generation of gravitational radiation

The binary pulsar data obtained by Taylor and coworkers (Taylor {39,40,41}) have already ruled out many alternative gravitation theories . Will {49} has adopted the post-Newtonian gravitational-radiation methods , developed within general relativity by Epstein and Wagoner { 9 } and by Wagoner and Will {48}, to alternative metric theories of gravitation, and thus has shown that in most (if not all) of these dipole gravitational radiation can also exist. The dipole radiation of a two body system, say, is generated by the varying dipole moment of the gravitational binding energy and is typically to large for the binary pulsar PSR 1913+16 . There are examples (e.g. the Rosen bimetric theory) where the dipole gravitational radiation causes the system even to gain energy at a relatively high rate and strong dissipative mechanisms would have to be invented to account for the observed decrease.

In this section we adapt the method of Epstein and Wagoner and show that the post-Newtonian generation of gravitational radiation is , for arbitrary λ , the same as in general relativity.

First we have to bring the field equations (7.23) and (7.24) into a convenient form by separating explicitly the linear terms in the fields $\phi_{\alpha\beta}$ and $a_{\alpha\beta}$. The term proportional to $(\lambda - 1)$ in (7.23) contains no linear terms and the field $a_{\alpha\beta}$ does not appear in the linearized part of ε_E^α. Hence we only have to split off the linear part in the Einstein form

$$\varepsilon_E^\alpha = \varepsilon_L^\alpha + \varepsilon_Q^\alpha \tag{7.44}$$

The linear part , ε_L^α, is given by

$$\varepsilon_L^\alpha = -G_L^{\alpha\beta}\,\eta_\beta \tag{7.45}$$

where $G_L^{\alpha\beta}$ is the linearized Einstein tensor

$$-G_L^{\alpha}{}_{\beta} = \Box(\phi^{\alpha}{}_{\beta} - \delta^{\alpha}{}_{\beta}\phi^{\gamma}{}_{\gamma}) + \delta^{\alpha}{}_{\beta}\phi^{\lambda\sigma}{}_{,\lambda\sigma} + \phi^{\lambda}{}_{\lambda}{}^{,\alpha}{}_{\beta} - (\phi^{\alpha\lambda}{}_{,\beta} + \phi^{\lambda}{}_{\beta}{}^{,\alpha})_{,\lambda} \tag{7.46}$$

The quadratic and higher order terms are easily obtained from (7.13):

$$\varepsilon_Q^\alpha = \Delta\varepsilon_S^\alpha + \{A^{\beta\alpha} + B^{\beta\alpha} + C^{\beta\alpha}\}\,\eta_\beta \tag{7.47a}$$

Where

$$A^{\nu\mu} = 2\,{}^*\{\theta^\nu \wedge d\theta^\alpha \wedge {}^*(d\theta_\alpha \wedge \theta^\mu)\} \tag{7.47b}$$

$$B^{\nu\mu} = 1/2\,{}^*\{\theta^\nu \wedge d\theta^\alpha \wedge {}^*d(\theta^\mu \wedge \theta_\alpha) + \theta^\mu \wedge d\theta^\alpha \wedge {}^*d(\theta^\nu \wedge \theta_\alpha)\} \tag{7.47c}$$

$$\begin{aligned} C^{\nu\mu} = -1/2\,{}^*\{&\theta^\nu \wedge \theta^\alpha \wedge d{}^*(d\theta_\alpha \wedge \Phi^\mu{}_\lambda dx^\lambda) + \\ &+ (dx^\nu \wedge \Phi^\alpha{}_\sigma dx^\sigma + \Phi^\nu{}_\lambda dx^\lambda \wedge dx^\alpha + \Phi^\nu{}_\lambda \Phi^\alpha{}_\sigma dx^\lambda \wedge dx^\sigma) \wedge d{}^*(d\theta_\alpha \wedge dx^\mu) \\ &+ \theta^\mu \wedge \theta^\alpha \wedge d{}^*(d\theta_\alpha \wedge \Phi^\nu{}_\lambda dx^\lambda) + \\ &+ (dx^\mu \wedge \Phi^\alpha{}_\lambda dx^\lambda + \Phi^\mu{}_\sigma dx^\sigma \wedge dx^\alpha + \Phi^\mu{}_\sigma \Phi^\alpha{}_\lambda dx^\sigma \wedge dx^\lambda) \wedge d{}^*(d\theta_\alpha \wedge dx^\nu)\} \end{aligned} \tag{7.47d}$$

The decomposition (7.47) is the analogon to equation (31) of Ref.{ 9 } and is quite useful for practical (post-Newtonian) calculations.

Inserting (7.44) into (7.23) gives

$$G_L^{\alpha\beta} = T_{eff}^{\alpha\beta} \tag{7.48}$$

with

$$T_{eff}^{\alpha\beta}\ \eta_\beta = t^\alpha + \varepsilon_Q^\alpha + (\lambda - 1)\ \Delta\varepsilon_S^\alpha \tag{7.49}$$

where $\Delta\varepsilon_S^\alpha$ is given by (7.20) and contains no linear terms. The last term in (7.49) is absent in general relativity.

We also write (7.24) for $\lambda \neq 1$ in a similar way :

$$\Box a^{\alpha\beta} + a^{\lambda\alpha,\beta}{}_\lambda - a^{\lambda\beta,\alpha}{}_\lambda = A_{eff}^{\alpha\beta} \tag{7.50}$$

where $A_{eff}^{\alpha\beta}$ collects the quadratic and higher order terms of

$$1/2\ d\{\theta^\alpha \wedge \theta^\beta \wedge *(d\theta^\gamma \wedge \theta_\gamma)\}$$

For outgoing wave-boundary conditions, we obtain from (7.48) and (7.49) the integral equations

$$\phi_{\alpha\beta}(t,\underline{x}) = \frac{-1}{4\pi} \int \frac{T_{\alpha\beta}(t-|\underline{x}-\underline{x}'|\ ,\underline{x}')}{|\underline{x} - \underline{x}'|}\ d^3x' + \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{7.51}$$

$$\left(T^\alpha{}_\beta = T_{eff}{}^\alpha{}_\beta - 1/2\ \delta^\alpha{}_\beta\ T_{eff}{}^\lambda{}_\lambda \right)$$

$$a_{\alpha\beta}(t,\underline{x}) = \frac{-1}{4\pi} \int \frac{A_{\alpha\beta}^{eff}(t\ -|\underline{x}-\underline{x}'|\ ,\underline{x}')}{|\underline{x} - \underline{x}'|}\ d^3x' + \eta_{\alpha,\beta} - \eta_{\beta,\alpha} \tag{7.52}$$

The gauge terms added on the right hand sides of (7.51) and (7.52) are, of course , not determined.

Let us now consider the fields $\phi_{\alpha\beta}$ and $a_{\alpha\beta}$ far from the source ($r = |\underline{x}| >> R$ = "size" of source, $|\phi_{\alpha\beta}| << 1$, $|a_{\alpha\beta}| << 1$) where the radiation is detected. In the approximation scheme of Ref.{ 9 }, one first expands the 1/r- terms with respect to the retardation within the source (slow motion).

In the next step, one constructs a post-Newtonian expansion of the sources $T_{eff}^{\alpha\beta}$ and $A_{eff}^{\alpha\beta}$. The necessary orders in the post-Newtonian expansion parameter, which are needed for the various

pieces are described in { 9 } and will not be repeated. In this approximation, the effective sources are contained within the near-zone. We may then employ (7.51) and (7.52) for field points within this region as well, again in terms of a post-Newtonian expansion. But the near zone post-Newtonian expansion was already studied in section 7.2. To the necessary orders $a_{\alpha\beta}$ vanishes, and $\phi_{\alpha\beta}$ agrees with general relativity if we impose suitable gauge conditions. ($A^{\alpha\beta}_{eff}$ and the last term in (7.49) for $T^{\alpha\beta}_{eff}$ both vanish in the near-zone post-Newtonian approximation.) Hence, we conclude from (7.51) and (7.52) that far away from the source $a_{\alpha\beta}$ is just equal to a gauge term and that $\phi_{\alpha\beta}$ agrees also there with general relativity. But in this weak field region, the equation for the a-field decouples and has a separate gauge invariance. Thus the a-field can be gauged away.

Finally we notice that the modified energy-momentum forms $t^{\alpha}_{L-L} - (\lambda - 1)\,\Delta t^{\alpha}$ (see eq. (7.34)) do not change the gravitational energy loss, because Δt^{α} is cubic in $\phi_{\alpha\beta}$. This follows from (7.35) and from the fact that the factor $d\theta^{\gamma}\wedge\theta_{\gamma}$ in (7.35) is already quadratic in $\phi_{\alpha\beta}$. (The linear term vanishes identically.) Taken all together proves our claim at the beginning of this section.

7.5. Conclusions

The main results of this chapter are the facts that not only the post-Newtonian approximation is independent of λ but that even the post-Newtonian generation of gravitational waves (developed within general relativity by Epstein and Wagoner{ 9 }) is the same as in general relativity. In particular, the "dipole catastrophe", described in the introduction of section 7.4. , which occurs in many alternative metric theories of gravitation, is absent. Therefore, the results obtained in { 9 } and {48} also hold in the theory by Hehl et al. At the present time, this theory can thus not be distinguished observationally from general relativity.

This is,of course, only true if the approximation scheme of Ref.{ 9 } is numerically reliable. Various authors (see e.g. { 8 } and references therein) have criticized the presently existing approximation methods for treating the radiation problem. We are aware of the critical questions that have been raised , but we think that the method used here is physically plausible.

Apart from aesthetic arguments, we see no way how to favour the Lagrangian with $\lambda = 1$ in (7.11),i.e. general relativity.

VIII Higher Dimensional Unification of Gauge Theory and Gravitation

General relativity describes gravity in a geometrical way, i.e. in terms of the metric field of the space-time continuum. In the second part of his life, Einstein tried to incorporate the electro-magnetism. His aim was a geometrical unification of the gravitational and the electro-magnetic field.

One of the most interesting attempts in this direction was the five-dimensional unification of gravitation and electro-magnetism by Klein and Kaluza {23,24}. This theory is briefly presented in section 8.1. The generalization of the Klein-Kaluza theory to non-abelian groups is discussed in section 8.2. The result is an elegant higher dimensional unification of non-abelian gauge theories with general relativity. In section 8.3. we incorporate a Higgs field and show that a slight modification of the generalized Klein-Kaluza metric leads to a symmetry breaking mechanism, provided that we admit $L_+^\uparrow$ as an additional gauge group.

8.1. Heuristic considerations

Let (M,g) be a Lorentz manifold and $(M,\mathcal{Q}_{max})$ a principal G-fibre bundle on M. For pedagogical reasons we assume M to be an element of $\mathcal{Q}_{max}$. (The general case will be discussed in the next section.) Then the product manifold

$$P(M,G) = M \times G \tag{8.1}$$

is called the bundle space associated with $(M,\mathcal{Q}_{max})$. The structure group G operates from the right on P(M,G) by

$$R_a(x,g) = (x,\bar{a}^{-1}.g) \quad , \quad a\varepsilon G \tag{8.2}$$

Obviously one has

$$P(M,G)/_G \simeq M \tag{8.3}$$

Let $\Pi: P(M,G) \longrightarrow M$ denote the canonical projection. Then for any $x \in M$ the set of points

$$\Pi^{-1}(x) = \{(x,g) \mid g \in G\}$$

is called the fibre over $x \in M$.

Any local gauge transformation $g: M \longrightarrow G$ defines an embedding i_g of M in P(M,G) by

$$i_g(x) = (x, g(x)) \tag{8.4}$$

Clearly $$\Pi . i_g = id_M \quad \text{for any } g(x). \tag{8.5}$$

Let A be a connection on $(M, \mathcal{O}_{max})$ and F the associated curvature. On the element $M \in \mathcal{O}_{max}$ the gauge potential defines a 1-form $A_M(x)$. The analogue holds for F and for any tensor p-form Ψ.

Now we lift these fields on the bundle space by defining

$$\tilde{A}(x,g) = gA_M(x)g^{-1} - dg.\, g^{-1} \tag{8.6}$$

$$\tilde{F}(x,g) = gF_M(x)g^{-1} \tag{8.7}$$

$$\tilde{\Psi}(x,g) = S(g).\Psi_M(x) \tag{8.8}$$

The term dg is formally to be understood as a coordinate differential (like dx^μ). This way of writing is not rigorous but suggestive. Obviously, $\tilde{A}(x,g)$ defines a 1-form, $\tilde{F}(x,g)$ a 2-form and $\tilde{\Psi}(x,g)$ a p-form on P(M,G). All the vectors tangent to a fibre $\Pi^{-1}(x) \equiv G$ are annihilated by (8.7) and (8.8), and hence these forms do not "measure" the g-coordinate on P(M,G). This is not true for (8.6). But the physically observable fields are always of the type (8.7) and (8.8).

One can see immediately that for any $a \in G$

$$(R_a^* \tilde{A})(x,g) = \bar{a}^{-1} \tilde{A}(x,g)\, a \tag{8.9}$$

$$(R_a^* \tilde{F})(x,g) = \bar{a}^{-1} \tilde{F}(x,g)\, a \tag{8.10}$$

$$(R_a^* \tilde{\Psi})(x,g) = S(\bar{a}^{-1})\, \tilde{\Psi}(x,g) \tag{8.11}$$

For any embedding $i_g : M \longrightarrow P(M,G)$ we may pull back the fields (8.6), (8.7), and (8.8) on M

$$(i_g^* \tilde{A})(x) = g(x) A_M(x) \bar{g}^{-1}(x) - dg(x).\bar{g}^{-1}(x) \tag{8.12}$$

$$(i_g^* \tilde{F})(x) = g(x) F_M(x) \bar{g}^{-1}(x) \tag{8.13}$$

$$(i_g^* \tilde{\Psi})(x) = S(g(x))\, \Psi_M(x) \tag{8.14}$$

This means that any embedding i_g of M in P(M,G) corresponds uniquely to a gauge on M.

Let $\mathcal{L}(x) = \mathcal{L}(F_{\mu\nu}(x), \Psi(x), D_\mu \Psi(x))$ be a locally gauge invariant Lagrange function. We lift $\mathcal{L}$ to a function $\tilde{\mathcal{L}}$ on the bundle space :

$$\tilde{\mathcal{L}}(x,g) = \mathcal{L}(x) + X^\mu(x,g)_{,\mu} \tag{8.15}$$

The divergence in (8.15) is simply a consequence of having substituted $\tilde{F}_{\mu\nu}$ for $F_{\mu\nu}$ etc. in $\mathcal{L}(x)$. Obviously, $\tilde{\mathcal{L}}$ in (8.15) is G-invariant. Thus the g-coordinate is just spurious and does not enter the field equations (on the bundle space) in a dynamical way. As a consequence of the local gauge invariance of physics we cannot "see" the higher dimensions.

Conversely one could be tempted to make use of these additional dimensions by extending the Lorentz metric $g_{\mu\nu}$ on M to a dynamical metric H_{ab} on M x G . We have seen in chapter V that the Lorentz frames θ^ν (and hence the coefficients of the metric) may be considered as gauge potentials.

It is almost compelling to construct a metric on P(M,G) such that the coefficients H_{ab} collect the dynamical gauge potentials of gravity (i.e. $g_{\mu\nu}$) and all the other dynamical gauge potentials in concern (Photon, $W^{\pm}$,Z etc.).

The first such attempt goes back to Klein and Kaluza {23,24} who managed to "unify" general relativity with Maxwell's theory.

They give the following metric on M x U(1)

$$H_{ab} = \left(\begin{array}{c|cccc} 1 & A_0 & A_1 & A_2 & A_3 \\ \hline A_0 & & & & \\ A_1 & & & & \\ A_2 & & g_{\mu\nu} + A_\mu A_\nu & & \\ A_3 & & & & \end{array}\right) \qquad (8.16)$$

Like a miracle the Ricci curvature ${}^{LC}_{R}(H_{ab})$ turns out to be the Lagrangian function for the coupled Einstein-Maxwell equations:

$${}^{LC}_{R}(H_{ab}) = {}^{LC}_{R}(g_{\mu\nu}) - 1/4\ F_{\mu\nu}F^{\mu\nu} \qquad (8.17)$$

8.2. Generalization of the Klein-Kaluza theory for non-abelian groups

The astonishing result of Klein and Kaluza has later inspired a lot of authors (among them Einstein and Pauli {1,33}). Generalizations to non-abelian gauge fields have been studied by de Witt{52}, Trautman {45} and Kerner {25}. A modern presentation in the language of principal fibre bundles was given by Cho{6}, but in order to understand this more sophisticated point of view , we need the full bundle geometry as developed in Ref.{27}.

Let $P(M,G_1),\ldots,P(M,G_n)$ be a finite squence of principal fibre bundles over a Lorentz manifold (M,g). By $\Pi_j : P(M,G_j) \longrightarrow M$ we denote the respective canonical projections.

This sequence of bundles admits a trivial unification with the bundle space

$$\tilde{P}(M,G_1 x \ldots x G_n) = \bigcup_{x\in M} \bar{\Pi}_1^{-1}(x) x \ldots x \bar{\Pi}_n^{-1}(x) \tag{8.18}$$

For any $j\varepsilon\{1,\ldots,n\}$ we have a canonical projection $p_j:\tilde{P} \longrightarrow P(M,G_j)$. Furthermore any sequence $\Psi_1,\ldots,\Psi_n$ of local trivialisations $\Psi_j:\bar{\Pi}_j^{-1}(U) \longrightarrow U \times G_j$ immediately yields a local trivialisation $\Psi: \bar{\Pi}^{-1}(U) \longrightarrow U \times G_1 \times \ldots \times G_n$ of $\tilde{P}$ and so on.

Given for each $j\varepsilon\{1,\ldots,n\}$ a connection A_j on $P(M,G_j)$ we can (canonically) construct a connection A on $\tilde{P}$ by

$$A = p_1^* A_1 \oplus \ldots \oplus p_n^* A_n \tag{8.19}$$

The curvature is

$$F = p_1^* F_1 \oplus \ldots \oplus p_n^* F_n \tag{8.20}$$

If Ψ is a section in a vector bundle associated with $P(M,G_{j_o})$ for some $j_o\varepsilon\{1,\ldots,n\}$, we can interpret Ψ as a section associated with $\tilde{P}$. The factors G_j , $j \neq j_o$, act trivially on Ψ .

Any compact group H admits a unique decomposition

$$H = H_1 \times \ldots \times H_m \times U(1) \times \ldots \times U(1) \tag{8.21}$$

where H_i is compact and semisimple. If $\mathfrak{h}_i$ denotes the Lie algebra of H_i then

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_m \oplus \ldots \oplus \mathbb{R} \tag{8.22}$$

Let k_i denote the Killing form on $\mathfrak{h}_i$. We define a <u>positive definite "Killing form " K</u> on $\mathfrak{h}$ by

$$K = k_1 \oplus \ldots \oplus k_m \oplus \ldots \oplus 1 \tag{8.23}$$

Now we have the tools to give the generalization of the Klein-Kaluza theory. The groups G_j in $\tilde{P}$ are supposed to be compact. The associated positive definite Killing forms are denoted by K_j. The generalized Klein-Kaluze metric on $\tilde{P}$ is

$$H(.,.) = \Pi^* g(.,.) + K_1(A_1(.),A_1(.)) + \ldots + K_n(A_n(.),A_n(.)) \qquad (8.24)$$

With the help of the straightforward technique presented in Ref.{28} it is relatively easy to find the following result for the Ricci curvature scalar

$$\begin{aligned} {}^{LC}R(H) &= \Pi^* \mathcal{L}_{K-K} \\ \mathcal{L}_{K-K} &= {}^{LC}R(g_{\mu\nu}) - 1/4\ F_1^2 - \ldots - 1/4\ F_n^2 + \\ &\quad + R_{G_1} + \ldots + R_{G_n} \end{aligned} \qquad (8.25)$$

Notice that $H(.,.)$ is G-invariant and ${}^{LC}R(H)$ is constant on the fibres. The constants $R_{G_j} \geq 0$ denote for every j the group curvature on G_j . (Any Lie group endowed with its left-invariant metric is an Einstein space of constant curvature $R_G = 1/4\ f^{abc} \ldots f_{abc}$; here $f^a{}_{bc}$ are the structure constants.)

F_j^2 stands for $\sum_{a,\mu,\nu} F_j^a{}_{\mu\nu} F_j^{a\,\mu\nu}$.

As expected the generalization (8.25) of (8.17) is now the Lagrangian function for the coupled Einstein-YM equations.

The sum $\Lambda = R_{G_1} + \ldots + R_{G_n}$ corresponds to a cosmological constant.

8.3. Prospects and open questions

The recent success of the Glashow-Salam-Weinberg model has proved that the YM-framework provides a promising theoretical basis for the description of the fundamental interactions of nature. All these models are , however, afflicted with a mysterious symmetry breaking mechanism: The Higgs boson Φ and the Higgs potential $V(\Phi)$ are both introduced by hand in order to endow some of the gauge fields (and matter fields) with masses.

It would be enlightening if a generalization of the G-invariant metric (8.24) would lead to a modification of (8.25) in the form

$$\mathcal{L} = {}^{LC}\!R(g_{\mu\nu}) - 1/4\ F_1^2 - .. - 1/4\ F_n^2 - 1/2\ D_\mu\Phi.D^\mu\Phi - V(\Phi) \tag{8.26}$$

where

$$V(\Phi) = A.\Phi^4 - B.\Phi^2 \quad , A > 0 \ , B > 0 \tag{8.27}$$

For instance , one could be tempted to choose for Φ a field on M , which takes its values in the space of the endomorphisms of the Lie algebra $\mathcal{G} = \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_n$ and which commutes with G , i.e.

$$Ad(a).\Phi.Ad(a^{-1}) = \Phi \quad ; \quad a\varepsilon G \tag{8.28}$$

The modified metric

$$\tilde{\tilde{H}}(.,.) = \Pi^*g + K(\Phi.A(.),\Phi.A(.)) \tag{8.29}$$

is still G-invariant. The associated curvature looks, however, awfully cumbersome and is far from being a realization of (8.26). Models of this type have been considered by Cho et al.{ 5 } and by Chang et al.{ 4 }.

A much simpler approach , which leaves the multiplet structure of Φ open, is the following :

We consider two gauge groups G_1 , G_2 and make the "Ansatz"

$$\tilde{H}(.,.) = \varepsilon^{-1}\ \Pi^*g(.,.) + \frac{\alpha}{\Phi^2} K_1(A_1(.),A_1(.)) + \beta K_2(A_2(.),A_2(.)) \tag{8.30}$$

The (positive) constants ε and α are dimensionless and β has the dimension of a $(\text{mass})^{-2}$ .The following result will be derived in appendix V :

$$\Phi^2\ {}^{LC}\!\tilde{R}(\tilde{H}) = \varepsilon\Phi^2\ {}^{LC}\!R(g_{\mu\nu}) - \frac{\alpha\varepsilon^2}{4} F_1^2 - \frac{\beta\varepsilon^2}{4}\Phi^2 F_2^2 - \frac{a}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi) \tag{8.31}$$

where $a > 0$ and $- V(\Phi) = 1/\alpha\ R_{G_1}\Phi^4 + 1/\beta\ R_{G_2}\Phi^2$ (8.32)

(By a minimal coupling $\partial_\mu\Phi \longrightarrow D_\mu\Phi$ one obtains a mass matrix for the gauge fields.)

There is no breaking mechanism if both , G_1 and G_2, are (semi-simple) compact groups . Hence we have to choose for G_1 a non-compact group . The only candidate, which avoids the problem of negative energies, is $L_+^\uparrow$ respectively its universal covering group $SL(2,\mathbb{C})$. Then the YM-type Lagrangian (6.18) replaces $-1/4\ F_1^2$, i.e. the total gravitational Lagrangian becomes

$$1/2\ \varepsilon\ \Phi^2\ \eta_{\mu\nu}\wedge^{LC}_{\Omega}{}^{\mu\nu} - \frac{\alpha\varepsilon^2}{2}\ \Omega^{\alpha}{}_{\beta}\wedge^{*}\Omega_{\alpha}{}^{\beta} \tag{8.33}$$

The Higgs potential then modifies to

$$V(\Phi) = 1/_{\alpha}\ A\ \Phi^4 - 1/_{\beta}\ B\ \Phi^2 \tag{8.34}$$

$$A = |R_{G_1}| > 0 \quad , \quad B = R_{G_2} > 0 \tag{8.35}$$

The potential (8.34) is minimized for

$$\Phi = v = \left(\frac{\alpha B}{2\beta A}\right)^{1/2} \tag{8.36}$$

and there takes the value

$$V(v) = -\frac{\alpha B^2}{4\beta^2 A} \tag{8.37}$$

In the broken symmetric phase the term $\Lambda = |V(v)|$ appears as a cosmological constant. The observational upper limit for Λ gives a constraint for α and β.

In the broken symmetric phase the Lagrangian

$$L = \{\varepsilon\ \Phi^2\ {}^{LC}R(g_{\mu\nu}) - 1/2\ (\partial_\mu\Phi,\partial^\mu\Phi) - V(\Phi)\}\ \eta + L_m \tag{8.38}$$

reduces to general relativity.(In the broken symmetric phase the Newtonian gravitation constant is $G = 1/\varepsilon v^2$). The Lagrangian (8.38) was recently proposed (ad hoc) by Zee{56}. We do ,however, not repeat the comprehensive discussion of Ref.{56}.

The total gravitational Lagrangian (8.33) satisfies the strong local Lorentz invariance (see section 4.2.). In the case

of Newtonian sources one would have to drop the YM-type part and vary the rest only with respect to the Lorentz frames. (See also section 7.1.) The resulting Lagrangian would be given by (8.38) with L_m describing Newtonian matter.

Clearly, this approach is not terribly profound and cannot seriously be considered as a geometrical foundation of the Higgs mechanism. Nevertheless, we think that the Higgs sector should admit a geometrical description, provided that Φ is really a new fundamental field. But this last point is an open question.

Appendix I

a) t_μ and $s^{\mu\nu}$ for matter fields

Let $L(\Psi, D\Psi)$ be some matter Lagrangian. Without loss of generality we assume that Ψ is a tensor o-form of type (S,E). Then we can recast the Lagrangian into the following form

$$L(\Psi, D\Psi) = \mathcal{L}(\Psi, D_\mu\Psi).\eta \tag{I.1}$$

where $D_\mu\Psi$ is defined by

$$D\Psi\wedge\eta_\mu = D_\mu\Psi.\eta \ , \text{ respectively } D\Psi = (D_\mu\Psi)\theta^\mu \tag{I.2}$$

A variation $\delta\theta^\mu$ does not affect Ψ and $D\Psi$. Hence we have

$$\delta(D_\mu\Psi.\theta^\mu) = \delta(D_\mu\Psi).\theta^\mu + D_\mu\Psi\ \delta\theta^\mu = 0 \tag{I.3}$$

and this implies

$$\delta\mathcal{L}.\eta = \frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ \delta(D_\mu\Psi).\eta = \frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ \delta(D_\nu\Psi)\delta^\nu_\mu\eta =$$

$$= \frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ \delta(D_\nu\Psi)\ \theta^\nu\wedge\eta_\mu = -\frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ D_\nu\Psi.\delta\theta^\nu\wedge\eta_\mu \tag{I.4}$$

Since $\delta\eta = \delta\theta^\mu\wedge\eta_\mu$ we finally conclude from

$$\delta L = \mathcal{L}\cdot\delta\theta^\mu\wedge\eta_\mu - \frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ D_\nu\Psi.\delta\theta^\nu\wedge\eta_\mu$$

that

$$t_\nu = (\delta^\mu_\nu\mathcal{L} - \frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ D_\nu\Psi) \tag{I.5}$$

$$t^\mu{}_\nu = \frac{\delta\mathcal{L}}{\delta D_\mu\Psi}\ D_\nu\Psi - \delta^\mu_\nu\mathcal{L} \tag{I.6}$$

Similarly $\delta_\omega(D\Psi) = 1/2\ \delta\omega_{\alpha\beta}\ S_\star(B^{\alpha\beta})\Psi$ and

$\delta_\omega L = \frac{\delta L}{\delta D\Psi}\wedge\delta(D\Psi)$. This implies immediately

$$s^{\alpha\beta} = -\frac{\delta L}{\delta D\Psi}S_\star(B^{\alpha\beta})\Psi \tag{I.7}$$

b) Isentropic perfect fluids *

We consider an isentropic perfect fluid with particle number density n and internal energy per particle ε. The infinitesimal volume per particle is $dv = -1/n^2\, dn$. Since the entropy is conserved along the flow lines, we have

$$0 = d\varepsilon + p\, dv = d\varepsilon - (1/n^2)\, p\, dn \text{ , i.e.}$$

$$p = n^2 \frac{d\varepsilon}{dn} \tag{I.8}$$

Let us assume that the particles of the fluid have an average mass $m \equiv 1$. Then the total energy density is

$$\mu = n(1 + \varepsilon) \tag{I.9}$$

We conclude

$$n \frac{d\mu}{dn} = (\mu + p) \tag{I.10}$$

$$n \frac{\partial \mu'}{\partial x^\alpha} = \frac{\partial p}{\partial x^\alpha} \qquad (\mu' = \frac{d\mu}{dn}) \tag{I.11}$$

The particle number is supposed to be conserved, i.e. there is a timelike 4-vector V^μ such that

(i) $$V^\mu V_\mu = 1 \tag{I.12}$$

(ii) $$(nV^\mu)_{|\mu} = 0 \tag{I.13}$$

We define the current 3-form j of the fluid by

$$j = j^\mu \eta_\mu = n\, V^\mu \eta_\mu \tag{I.14}$$

Then (13) may be rewritten as

$$\mathcal{L}_V j = dj = 0 \tag{I.15}$$

Variations of V

Let D be a connected compact region with regular boundary. On D we generate variations of V by applying vector fields K with $\mathrm{supp}(K) \subset D$. The flow $\Phi(x,s)$ associated with K is complete and $\Phi_s D = D$ for any $s \in R$.

* We follow the presentation in Ref. [15].

We define for $(x,s)\in M \times \mathbb{R}$

$$\bar{V}(x,s) = (\Phi_{-s})_{*\Phi(x,s)} \cdot V(\Phi(x,s))$$

Since $g(\bar{V}(x,o),\bar{V}(x,o)) = 1$ and since D is connected and compact, there exists an $\varepsilon > 0$, such that

$$g(\bar{V}(x,s),\bar{V}(x,s)) > 0 \quad \text{for all } (x,s)\in D \times [-\varepsilon,\varepsilon] .$$

For such arguments we define

$$\tilde{V}(x,s) = \frac{\bar{V}(x,s)}{\{g(\bar{V}(x,s),\bar{V}(x,s))\}^{\frac{1}{2}}} .$$

Actually $\tilde{V}(x,s)$ is the normalized tangent vector of the deviated flow lines. From

$$\frac{\partial}{\partial s}\Big|_o \bar{V}(x,s) = {}^{LC}\nabla_K V - {}^{LC}\nabla_V K \quad \text{and} \quad g(V,{}^{LC}\nabla_K V) = 0$$

we find immediately the following expression for the variation of V :

$$\delta V^\mu = V^\mu{}_{|\nu} K^\nu - K^\mu{}_{|\nu} V^\nu + V^\alpha V^\beta K_{\alpha|\beta} V^\mu \tag{I.16}$$

Variations of n

The Lagrangian is taken to be

$$L = -\mu.\eta = -n(1+\varepsilon)\eta \tag{I.17}$$

The action $I = \int_D L$ is required to be stationary when the flow lines are varied. The density is adjusted to keep j conserved. This means for the continuity equation

$$(\delta n.V^\mu)_{|\mu} + (n.\delta V^\mu)_{|\mu} = 0 \tag{I.18}$$

By equation (13) and (16) this may be rewritten as

$$n\{(\frac{\delta n}{n}) + V^\alpha V^\beta K_{\alpha|\beta}\}_{|\mu} V^\mu + \{n(V^\mu{}_{|\nu} K^\nu - K^\mu{}_{|\nu} V^\nu)\}_{|\mu} = 0 \tag{I.19}$$

Some straightforward computation shows that

$$\{n(V^\mu{}_{|\nu} K^\nu - K^\mu{}_{|\nu} V^\nu)\}_{|\mu} = -n L_V\{\frac{1}{n}(nK^\mu)_{|\mu}\}$$

Combined with (19) this implies

$$L_V\{(\frac{\delta n}{n}) + V^\alpha V^\beta K_{\alpha|\beta} - \frac{1}{n}(n K^\mu)_{|\mu}\} = 0 \qquad (I.20)$$

But equation (20) says that the bracket term is constant along the flow lines of V. Since it vanishes on ∂D , we have

$$\delta n = (nK^\nu)_{|\nu} - n V^\alpha V^\beta K_{\alpha|\beta} \qquad (I.21)$$

The equations of motion resulting from $\delta I = 0$ and from (21) are

$$(\mu + p)\dot{V}^\alpha = p_{|\beta}(g^{\alpha\beta} - V^\alpha V^\beta) \qquad (I.22)$$

$$(\dot{V}^\alpha = V^\alpha{}_{|\beta} V^\beta)$$

The energy-stress tensor

The energy-stress tensor is as usually defined by

$$\delta L = \delta\theta^\mu \wedge t_\mu \qquad (I.23)$$

The variations of the frames do not affect the flow lines. Let Σ be a spacelike surface. Then for any 3-volume $\Delta \subset \Sigma$ the amount of fluid in Δ is not affected by $\delta\theta^\mu$ either, i.e.

$$\int_\Delta \delta j = 0$$

for all such Δ and this implies

$$\delta j = 0 \qquad (I.24)$$

From $n^2.\eta = *j\wedge j$ we have with (24)

$$2n\delta n\,\eta + n^2\delta\eta = \delta(*j)\wedge j$$

Furthermore $\theta^\alpha\wedge j = *j\wedge\eta^\alpha$ and thus

$$\delta(*j)\wedge\eta^\alpha + *j\wedge\delta\eta^\alpha = \delta\theta^\alpha\wedge j$$

We contract this last equation with j_α and find

$$\delta(*j)\wedge j = \delta\theta^\alpha\wedge\{2j_\alpha.j - n^2.j_\alpha\} = 2n.\delta n.\eta + n^2\delta\eta$$

respectively

$$\delta n.\eta = \delta\theta^{\alpha}\wedge(V_{\alpha}.j - n\eta_{\alpha}) \tag{I.25}$$

This implies with equation (8)

$$\delta\mu.\eta = \delta\theta^{\alpha}\wedge\{(\mu + p)(V_{\alpha}V^{\beta}.\eta_{\beta} - \eta_{\alpha})\}$$

and finally

$$t_{\alpha} = p\,\eta_{\alpha} - (\mu + p)\,V_{\alpha}V^{\beta}\,\eta_{\beta} \tag{I.26}$$

$$t_{\alpha\beta} = (\mu + p)\,V_{\alpha}V_{\beta} - p\,g_{\alpha\beta} \tag{I.27}$$

The spin-density

By the same argument as above we have for any variations $\delta\omega^{\alpha}{}_{\beta}$ of the connection : $\delta j = 0$. But since V^{μ} is not affected either, i.e. $\delta V^{\mu} = 0$, we conclude

$$0 = \delta j = \delta n\,V^{\mu}\,\eta_{\mu}$$

$\delta n = 0$ means, however, that the associated spin-density is equal to zero:

$$s^{\alpha\beta} = 0 \tag{I.28}$$

Appendix II

Let ξ_μ be a local Lorentz frame and $V = V^\mu \xi_\mu$ a vector field with compact support. The associated flow $\Phi(x,s)$ is complete and we may define an element $\Phi_V \in Diff(M)$ by

$$\Phi_V : M \longrightarrow M \quad : \quad x \longrightarrow \Phi(x,1)$$

For any $s \in \mathbb{R}$ one has

$$\Phi_{sV}(x) = \Phi(x,s) \tag{II.1}$$

and this implies the group property

$$\Phi_{sV} \cdot \Phi_{tV} = \Phi_{(s+t)V} \tag{II.2}$$

Now we may be tempted to interpret $V^\mu(x)$ as a local T^4-valued gauge transformation, which operates on tensor o-forms as

$$\left[T(V)\Psi\right](x) = (\Phi_V^* \Psi)_{(x)} = \Psi^\alpha{}_{(\Phi_V(x))} \, E_\alpha \tag{II.3}$$

This definition allows the following group property

$$\left[T(sV)\left[T(tV)\Psi\right]\right]_{(x)} = (\Phi_{sV}^* \, (\Phi_{tV}^* \Psi))_{(x)} =$$

$$= (\Phi_{(s+t)V}^* \Psi)_{(x)} = \left[T((s+t)V)\Psi\right]_{(x)}$$

Now we bring the Lorentz rotations into play. Let B be an element of so(1.3). Then we denote by

$$g_s(a,B) = \exp\{sa \, \Phi_f \, sB\} \tag{II.4}$$

the corresponding one-parameter subgroup of the Poincaré group.

We consider local so(1,3)-valued functions A(x) subject to the constraint $(\Phi_{sV}^* A)_{(x)} = A(\Phi_{sV}(x)) = A(x)$ for all $s \in \mathbb{R}$.

Then $g_s(V(x),A(x))$ admits the following representation

$$\left[g_s(V,A).\Psi\right]_{(x)} = S(e^{sA(x)}) \left[\Phi_{sV}^* \, \Psi\right]_{(x)} \tag{II.5}$$

This definition respects the group property

$$\left[g_s(V,A)\left[g_t(V,A).\Psi\right]\right]_{(x)} = \left[g_{s+t}(V,A).\Psi\right]_{(x)} \qquad \text{(II.6)}$$

The invariance of a Lagrangian with respect to such one-parameter groups $g_s(V(x),A(x))$ of local Poincaré transformations covers both, the local Lorentz invariance and the diffeomorphism invariance. This sort of thing is, however, a rather artificial one and does not deserve to be considered as an alternative way of gauging the Poincaré group.

The infinitesimal version of the above concept is the Poincaré symmetry as proposed by Hehl et al.{19}. A discussion of the Hehl-concept is given in section 2 of Ref.{38} and will not be repeated here.

Appendix III

With the help of (6.63) we bring the Hilbert-Einstein Lagrangian

$$L_E = 1/2\ \eta_{\mu\nu} \wedge {}^{LC}\Omega^{\mu\nu} \tag{III.1}$$

in the alternative form (6.48). Modulo an exact differential we have

$$1/2\ \eta_{\mu\nu} \wedge d{}^{LC}\omega^{\mu\nu} = -\ 1/4\ d\theta^{\sigma} \wedge (\delta^{\lambda}_{\sigma}\ \eta_{\mu\nu} + \delta^{\lambda}_{\mu}\eta_{\nu\sigma} + \delta^{\lambda}_{\nu}\eta_{\sigma\mu}) \cdot \cdot * \{d\theta^{\mu} \wedge \eta^{\nu}{}_{\lambda} - d\theta^{\nu} \wedge \eta^{\mu}{}_{\lambda} - d\theta_{\lambda} \wedge \eta^{\mu\nu}\} \tag{III.2}$$

With $d\theta^{\lambda} = -1/2\ G^{\lambda}{}_{\rho\sigma}\theta^{\rho} \wedge \theta^{\sigma}$ we find

$$d\theta^{\lambda} \wedge \eta_{\mu\nu} = -\ G^{\lambda}{}_{\mu\nu} \cdot \eta \tag{III.3}$$

Expressing (2) in terms of the $G^{\lambda}{}_{\mu\nu}$ yields

$$1/2\eta_{\mu\nu} \wedge d{}^{LC}\omega^{\mu\nu} = \{1/2\ G^{\lambda}{}_{\alpha\beta} G^{\alpha\beta}{}_{\lambda} - 1/4\ G_{\alpha\beta\gamma} G^{\alpha\beta\gamma} + G^{\rho}{}_{\rho\mu} G^{\sigma}{}_{\sigma}{}^{\mu}\}\ \eta \tag{III.4}$$

Similarly one finds immediately

$$1/2\ \eta_{\mu\nu} \wedge {}^{LC}\omega^{\mu}{}_{\alpha} \wedge {}^{LC}\omega^{\alpha\nu} = \{-1/2\ G^{\rho}{}_{\rho\mu} G^{\sigma}{}_{\sigma}{}^{\mu} + 1/4\ G_{\alpha\beta\lambda} G^{\beta\alpha\lambda} + 1/8\ G_{\alpha\beta\gamma} G^{\alpha\beta\gamma}\}\eta \tag{III.5}$$

The sum of (4) and (5) finally yields

$$1/2\eta_{\mu\nu} \wedge {}^{LC}\Omega^{\mu\nu} = -1/2\{1/4\ G_{\alpha\beta\gamma} G^{\alpha\beta\gamma} + 1/2\ G_{\alpha\beta\lambda} G^{\beta\alpha\lambda} - G^{\rho}{}_{\rho\mu} G^{\sigma}{}_{\sigma}{}^{\mu}\}\eta$$

$$= -1/2(d\theta^{\alpha} \wedge \theta^{\beta}) \wedge * (d\theta_{\beta} \wedge \theta_{\alpha}) + 1/4(d\theta^{\alpha} \wedge \theta_{\alpha}) \wedge * (d\theta^{\beta} \wedge \theta_{\beta}) \tag{III.6}$$

Appendix IV

In this appendix , we derive the expression (7.43) for the angular momentum. From the definition (7.39) and the field equations in the form (7.33) we conclude that

$$\sqrt{-g}\, M^{\sigma\alpha} = 1/2\ (x^{\sigma}dh^{\alpha} - x^{\alpha}dh^{\sigma}) = \tag{IV.1}$$

$$= 1/2\ d(x^{\sigma}h^{\alpha} - x^{\alpha}h^{\sigma}) - 1/2(dx^{\sigma}\wedge h^{\alpha} - dx^{\alpha}\wedge h^{\sigma})$$

where

$$h^{\alpha} = -\sqrt{-g}\ {}^{LC}\omega^{\beta\gamma}\wedge\eta^{\alpha}{}_{\beta\gamma} \tag{IV.2}$$

Now we write also the last term in (1) as an exact differential. We have

$$dx^{\sigma}\wedge h^{\alpha} - dx^{\alpha}\wedge h^{\sigma} = \sqrt{-g}\ {}^{LC}\omega^{\beta\gamma}\wedge dx^{\sigma}\wedge\eta^{\alpha}{}_{\beta\gamma} - (\alpha\leftrightarrow\sigma) =$$

$$= \sqrt{-g}\ ({}^{LC}\omega_{\beta}{}^{\sigma}\wedge\eta^{\alpha\beta} + {}^{LC}\omega^{\sigma}{}_{\beta}\wedge\eta^{\beta\alpha} - {}^{LC}\omega_{\beta}{}^{\alpha}\wedge\eta^{\sigma\beta} - {}^{LC}\omega^{\alpha}{}_{\beta}\wedge\eta^{\beta\sigma})$$

Here we use

$$d\eta^{\sigma\alpha} + {}^{LC}\omega^{\sigma}{}_{\beta}\wedge\eta^{\beta\alpha} + {}^{LC}\omega^{\alpha}{}_{\beta}\wedge\eta^{\sigma\beta} = 0$$

and obtain

$$dx^{\sigma}\wedge h^{\alpha} - dx^{\alpha}\wedge h^{\sigma} = \sqrt{-g}\{{}^{LC}\omega_{\beta}{}^{\sigma}\wedge\eta^{\alpha\beta} - (\sigma\leftrightarrow\alpha) - d\eta^{\sigma\alpha}\}$$

But $${}^{LC}\omega_{\beta}{}^{\sigma}\wedge\eta^{\alpha\beta} = \Gamma_{\beta\mu}^{\sigma}\, dx^{\mu}\wedge\eta^{\alpha\beta} = \Gamma_{\beta}^{\beta\sigma}\,\eta^{\alpha} - \Gamma_{\beta}^{\alpha\sigma}\,\eta^{\beta}$$

and hence

$$dx^{\sigma}\wedge h^{\alpha} - dx^{\alpha}\wedge h^{\sigma} = \sqrt{-g}\ \{\Gamma_{\beta}^{\beta\sigma}\,\eta^{\alpha} - \Gamma_{\alpha}^{\beta\alpha}\,\eta^{\sigma} - d\eta^{\sigma\alpha}\}$$

If we use in this expression

$$\Gamma_{\beta}^{\beta\sigma} = \frac{1}{\sqrt{-g}}\ g^{\mu\sigma}\,\partial_{\mu}\sqrt{-g}$$

then we find easily

$$dx^{\sigma}\wedge h^{\alpha} - dx^{\alpha}\wedge h^{\sigma} = -d\{\sqrt{-g}\ \eta^{\sigma\alpha}\} \tag{IV.3}$$

With this result and (2) equation (1) becomes

$$\sqrt{-g}\ M^{\sigma\alpha} = 1/2\ d\{\sqrt{-g}\ \{\eta^{\sigma\alpha} + (x^{\sigma}\eta^{\alpha}{}_{\beta\gamma} - x^{\alpha}\eta^{\sigma}{}_{\beta\gamma})\wedge{}^{LC}\omega^{\beta\gamma}\}\} \tag{IV.4}$$

Inserting this into (7.43) and using Stokes' theorem finally gives equation (7.43).

Appendix V

In order to derive (8.31) we start with the simplified metric

$$H(.,.) = \bar{\varepsilon}^{-1}\Pi^{*}g + \frac{\alpha}{\Phi^{2}} K_{1}(A_{1}(.),A_{1}(.)) \tag{V.1}$$

We put $\hat{g} = \Phi^{2}g$; $\Omega^{2} = \Phi^{-2}$; $N = \mathrm{Dim}(G_{1}) + 4$ (V.2)

Then (1) can be rewritten as

$$H(.,.) = \Omega^{2}\{\bar{\varepsilon}^{-1}\Pi^{*}\hat{g} + \alpha K_{1}(A_{1}(.),A_{1}(.))\} = \Omega^{2}\,\hat{H}(.,.)$$

By formula (2.30) of Ref.{15} we have

$$\overset{LC}{R}(H) = \Omega^{-2}\,\overset{LC}{R}(\hat{H}) - 2\,(N-1)\,\Omega^{-3}\,\Omega\hat{|}^{a}\,\hat{|}_{a}\cdot\varepsilon - \varepsilon(N-1)(N-4)\Omega_{,\mu}\Omega\hat{|}^{\mu}\Big/\Omega^{4} \tag{V.4}$$

$\overset{LC}{R}(\hat{H})$ is easily computed from (8.25) :

$$\overset{LC}{R}(\hat{H}) = \Pi^{*}\hat{\mathfrak{L}}$$

$$\hat{\mathfrak{L}} = \varepsilon\,\overset{LC}{R}(\hat{g}_{\mu\nu}) - \frac{\varepsilon^{2}\alpha}{4}\hat{F}_{1}^{2} + 1/\alpha\; R_{G_1} =$$

$$= \varepsilon\,(1/\Phi^{2})\,\overset{LC}{R}(g_{\mu\nu}) - \frac{\varepsilon^{2}\alpha}{4}\Phi^{-4}F_{1}^{2} + 1/\alpha\; R_{G_1} - 6\,\Phi^{-3}\,\Phi|^{\mu}{}_{|\mu}\cdot\varepsilon \tag{V.5}$$

Furthermore*)

$$\Omega\hat{|}^{a}\,\hat{|}_{a} = (\hat{g}^{\mu\nu}\Omega_{,\nu})\hat{|}_{\mu} = \frac{1}{\sqrt{|\hat{g}|}}\,(\sqrt{|\hat{g}|}\,(1/\Phi^{2})\,g^{\mu\nu}\Omega_{,\nu})_{,\mu} =$$

$$= -\frac{1}{\Phi^{4}}\,\Phi|^{\mu}{}_{|\mu} \qquad \text{and}$$

$$\Omega\hat{|}^{\mu} = -g^{\mu\nu}\frac{\Phi_{,\nu}}{\Phi^{4}} \tag{V.6}$$

---

*) At this point we make use of the fact that in a suitable frame on P(M,G) the Christoffelsymbols $\Gamma^{i}_{j\mu} = 0$, where $\mu = 0,1,2,3$ and $i,j = 1,\ldots,\mathrm{Dim}(G_{1})$, (See for this Ref.{28}.)

Taken all together yields ${}^{LC}_{R}(H) = \Pi^* \mathcal{L}$

$$\mathcal{L} = \varepsilon . {}^{LC}_{R}(g_{\mu\nu}) - \frac{\varepsilon^2\alpha}{4}\,\Phi^{-2}\,F_1^2 + 1/\alpha\; R_{G_1}\Phi^2 + \\ + 2(N-4)\varepsilon.\Phi|^{\mu}|_{\mu}\Big/\Phi \;-\; (N-1)(N-4)\varepsilon.\Phi\,|^{\mu}\Phi\,|^{\mu}\Big/\Phi^2 \qquad (V.7)$$

For the generalized metric (8.30) we now find immediately*):

$${}^{LC}_{R}(\tilde{H}) = \varepsilon\,{}^{LC}_{R}(g_{\mu\nu}) - \frac{\varepsilon^2\alpha}{4}\,\Phi^{-2}\,F_1^2 - \frac{\varepsilon^2\beta}{4}\,F_2^2 + \\ + \left(1/\alpha\; R_{G_1}\Phi^2 + 1/\beta\; R_{G_2}\right) + 2(N-4).\varepsilon.\Phi|^{\mu}|_{\mu}\Big/\Phi \;-\; (N-1)(N-4)\frac{\varepsilon\Phi_{,\mu}\Phi^{,\mu}}{\Phi^2}$$

respectively

$$\Phi^2\,{}^{LC}_{R}(\tilde{H}) = \Phi^2.\varepsilon.{}^{LC}_{R}(g_{\mu\nu}) - \frac{\varepsilon^2\alpha}{4}\,F_1^2 - \frac{\varepsilon^2\beta}{4}\,\Phi^2F_2^2 + \\ + \left(1/\alpha\; R_{G_1}\Phi^4 + 1/\beta\; R_{G_2}\Phi^2\right) + 2(N-4)\varepsilon\Phi\Phi|^{\mu}|_{\mu} - (N-1)(N-4)\varepsilon\Phi_{,\mu}\Phi^{,\mu}$$

$$\doteq \Phi^2.\varepsilon.{}^{LC}_{R}(g_{\mu\nu}) - \frac{\varepsilon^2\alpha}{4}\,F_1^2 - \frac{\varepsilon^2\beta}{4}\,\Phi^2F_2^2 + \\ + \left(1/\alpha\; R_{G_1}\Phi^4 + 1/\beta\; R_{G_2}\Phi^2\right) - \varepsilon(N+1)(N-4)\Phi_{,\mu}\Phi^{,\mu} \qquad (V.8)$$

In particular we have

$$a = 2(N+1)(N-4) > 0$$

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*) ${}^{LC}_{R}(\tilde{H})$ is now understood as a function on M.

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